Module 3 : Wave Model

Lecture 3.1 : Wave Model - I

Objectives

In this lecture you will learn the following

- **Analysis of cylindrical waveguides**

- **Characteristic Equation (Eigen Value Equation)**

- **Modes of optical fiber**

MODAL PROPAGATION INSIDE AN OPTICAL FIBER (WAVE MODEL-I)

1. Basics

1. There are certain limitations to ray model.

2. The ray model does not predict correctly that even after total internal reflection there will be some field in the cladding
Also it does not predict that rays can be launched at only discrete angles in an optical fiber.

3. For an accurate and complete description of light propagation inside an optical fiber we have to go in for a more
rigorous model, called the **WAVE MODEL.** Here we shall discuss the propagation of light inside an optical fiber, treating light as an electromagnetic wave.

4. Inside a fiber core the optical energy gets guided i.e. the energy propagates along the axis of the core and the fields
exponentially decay in the cladding away from the core-cladding interface.

5. In a practical fiber the cladding is surrounded by a protective layer. Generally, by the time the field reaches that layer, it

dies down significantly so that the protecting layer does not affect the propagation of the wave significantly. That is, the whole propagation of light is governed by the core cladding interface and the interface between the cladding and other protecting layers does not affect the propagation. In other words, we can take the cladding to be of infinite size in our analysis, without incurring significant error.

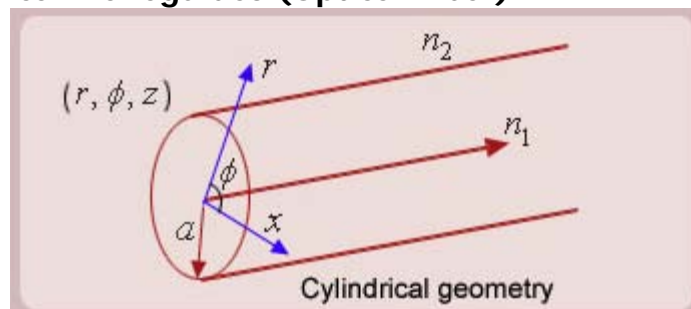**2. Analysis of cylindrical waveguides (Optical Fiber)**

Figure (1)

1. As shown in Figure 1, the core of the optical fiber is a cylinder of radius a, and of refractive index n_1. The refractive index of cladding is n_2 and the cladding is of infinite radius.

2. The appropriate coordinate system to analyze this problem is the cylindrical coordinate system, (r,ϕ,z). The wave propagates in the z direction and the fields have definite distributions in the cross sectional plane, defined by (r,ϕ). Any radial direction from the center of the fiber is denoted by r and the azimuthal angle measured from a reference axis (x-axis) in the cross-sectional plane is denoted by ϕ.

3. To investigate an electromagnetic problem we start with the Maxwell's equations. Here we investigate the propagation of light in the fiber without worrying about the origin of the light inside the fiber. In other words we assume that the Maxwell's equations which govern the electromagnetic radiations inside the fiber are **source free** .

4. Maxwell's equations for a source-free medium (i.e., the charge density and the conduction current densities in the medium are zero):

(a) $\nabla . D = 0$ **(1)**

(b) $\nabla . B = 0$ **(2)**

(c) $\nabla \times E = -\dfrac{\partial B}{\partial t}$ **(3)**

(d) $\nabla \times H = -\dfrac{\partial D}{\partial t}$ **(4)**

where D is the electric displacement vector, **B** is the magnetic flux density, **E** is the electric field, and **H** is the magnetic field intensity.

We have two more equations, called the constitutive relations, as

$B = \mu H$

$D = \varepsilon E$

where μ is the permeability of the medium and ε is the permittivity of the medium.

Now we have to decouple the equations (3) & (4).

For this, we take curl of equation (3) as

$$\nabla \times \nabla \times E = -\nabla \times \dfrac{\partial B}{\partial t}$$

Substituting for $B = \mu H$ and interchanging the space and time derivatives, we get

$$\nabla \times \nabla \times E = -\dfrac{\partial}{\partial t}\nabla \times (\mu H)$$

(Since we assume a homogeneous medium,   is not a function of space).

$$= -\mu \frac{\partial}{\partial t} (\nabla \times H)$$ 　　　　　　　　　μ

Now substituting from eqn.(4) we get

$$\nabla \times \nabla \times E = -\mu \frac{\partial}{\partial t} \left(\frac{\partial D}{\partial t} \right)$$

$$= -\mu \varepsilon \frac{\partial}{\partial t} \cdot \frac{\partial}{\partial t} \cdot E$$

$$\nabla \times \nabla \times E = -\mu \varepsilon \frac{\partial^2 E}{\partial t^2}$$

From vector algebra we have the identity

$$\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E$$

so the above equation can be simplified to

$$\nabla (\nabla \cdot E) - \nabla^2 E = -\mu \varepsilon \frac{\partial^2 E}{\partial t^2}$$

For a homogeneous medium ε is not a function of space. Eqn (1) then gives
$$\nabla \cdot D = \nabla \cdot (\varepsilon E) = \varepsilon \nabla \cdot E = 0 \cdot$$

$$\Rightarrow \nabla \cdot E = 0$$

Substituting in above eqn.

we get

$$\boxed{\nabla^2 E = \mu \varepsilon \frac{\partial^2 E}{\partial t^2}} \text{(5)}$$

This equation is called the **Wave Equation** .

If we do the similar analysis for the magnetic field we get the same wave equation for the magnetic field

$$\boxed{\nabla^2 H = \mu \varepsilon \frac{\partial^2 H}{\partial t^2}} \text{(6)}$$

To analyse the propagation of light inside an optical fiber, we have to solve the wave equation with appropriate boundary conditions.

5. Since we have chosen the cylindrical coordinate system (r, ϕ, z), we

write the wave equation for the electric field (and magnetic field) in the cylindrical

coordinates as:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E}{\partial \phi^2} + \frac{\partial^2 E}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E}{\partial t^2} \quad (7)$$

1. The electric field, E is a vector quantity having three components and the magnetic filed, H is a vector quantity

also having three components. So we have total 6 field components. However all these components are not independent of each other since $E \& H$ are related through the Maxwell's equations. We can take two components as independent components and express the remaining four components in terms of the independent components. Since in this case the wave propagates along the axis of the fiber i.e., in the z direction, generally the two components, $E_z \& H_z$ (also called the longitudinal components) are taken as independent components and the other four transverse field components i.e., E_r, E_ϕ, H_r, H_ϕ are expressed in terms of these two components.

The wave equation is solved for the two longitudinal components and the transverse components can be obtained by substitution of the longitudinal components in the Maxwell's equations.

7. (a) The transverse components are related to the longitudinal components as

$$E_r = \frac{-j}{q^2}\left\{ \beta\frac{\partial E_z}{\partial r} + \frac{\mu\omega}{r}\frac{\partial H_z}{\partial \phi} \right\}$$

$$E_\phi = \frac{-j}{q^2}\left\{ \frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \mu\omega\frac{\partial H_z}{\partial r} \right\}$$

$$H_r = \frac{-j}{q^2}\left\{ \beta\frac{\partial H_z}{\partial r} - \frac{\omega\varepsilon}{r}\frac{\partial E_z}{\partial \phi} \right\}$$

$$H_\phi = \frac{-j}{q^2}\left\{ \frac{\beta}{r}\frac{\partial H_z}{\partial \phi} + \omega\varepsilon\frac{\partial E_z}{\partial r} \right\}$$

where $q^2 = \omega^2\mu\varepsilon - \beta^2$ (β is the propagation constant of the wave along the axis of the fiber. This parameter is defined subsequently)

b) The wave equation is to be solved for E_z and H_z which are scalar quantities. So in general the wave equation is

solved for a scalar function ψ, where ψ represents either E_z or H_z.

Writing the above wave equation in terms of the scalar function ψ we get

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = \mu\varepsilon\frac{\partial^2 \psi}{\partial t^2} \quad \text{_____ (8)}$$

8. For a general solution of the wave equation we apply separation of variables. We assume the solution as

$$\psi = R(r)\Phi(\phi)Z(z)T(t)$$

Let us assume that the fields are time harmonic fields, that is, all electric and magnetic field components have time variation $e^{j\omega t}$. That is,

$$\psi \sim e^{j\omega t} \Rightarrow T(t) = e^{j\omega t}$$

$$\Rightarrow \frac{\partial}{\partial t} \equiv j\omega$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \equiv -\omega^2$$

Equation (8) then becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = -\omega^2 \mu\varepsilon\psi$$

a) Since the energy has to propagate along the axis of the fiber, the solution should be a traveling wave type solution

along the axis, that is, the z direction. If a wave travels in $+z$ direction then its z-variation should be $e^{-j\beta z}$. That is to say that

$$\psi \sim e^{-j\beta z} \Rightarrow Z(z) = e^{-j\beta z}$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} \equiv -\beta^2$$

b) Now to fix the ϕ variation we can use the following argument . From Figure1 we can note that, if we move only in the ϕ

direction i.e., in the azimuthal direction in a cross sectional plane, after one complete rotation we reach to the same location. In other words, the function ψ is periodic in ϕ over 2π. In ϕ direction, the function is a harmonic function that is,

$$\psi \sim e^{j\nu\phi} \Rightarrow \Phi(\phi) = e^{-j\nu\phi}$$

where ν is an integer. This functional form represents a field which will repeat itself after one rotation or when ϕ changes by multiples of 2π.

Substituting for the $\Phi(\phi), Z(z), T(t)$ in the wave equation the only unknown function remains to be evaluated is $R(r)$.

The wave equation therefore becomes

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left\{ -\frac{\nu^2}{r^2} - \beta^2 + \omega^2\mu\varepsilon \right\}R = 0 \quad \textbf{(9)}$$

Since $q^2 = \omega^2\mu\varepsilon - \beta^2$ as was defined earlier, the final wave equation will be

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(q^2 - \frac{\nu^2}{r^2}\right)R = 0 \underline{\hspace{2cm}} \textbf{(10)}$$

This equation is the **Bessel's equation** and solutions of this equation are called the **Bessel functions** .

Field variation in the transverse plane in the radial direction will be governed by the Bessel's equation and the field distributions would be Bessel functions.

9. We have a variety of solutions to the Bessel's equation depending upon the parameters

ν and q. ν is an integer and a positive quantity.
Depending upon the choice of q i.e., a) real, b) imaginary, c) complex,
we get different solutions to the Bessel's equation. So to choose the proper solution we must have the physical understanding of the field distribution.

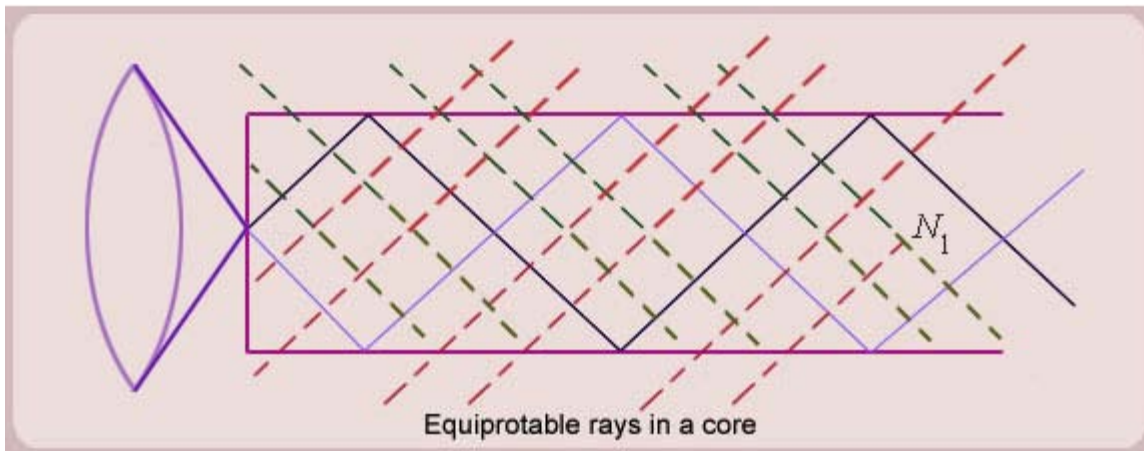Here we use the physical understandings gathered from the ray model of the light propagation.

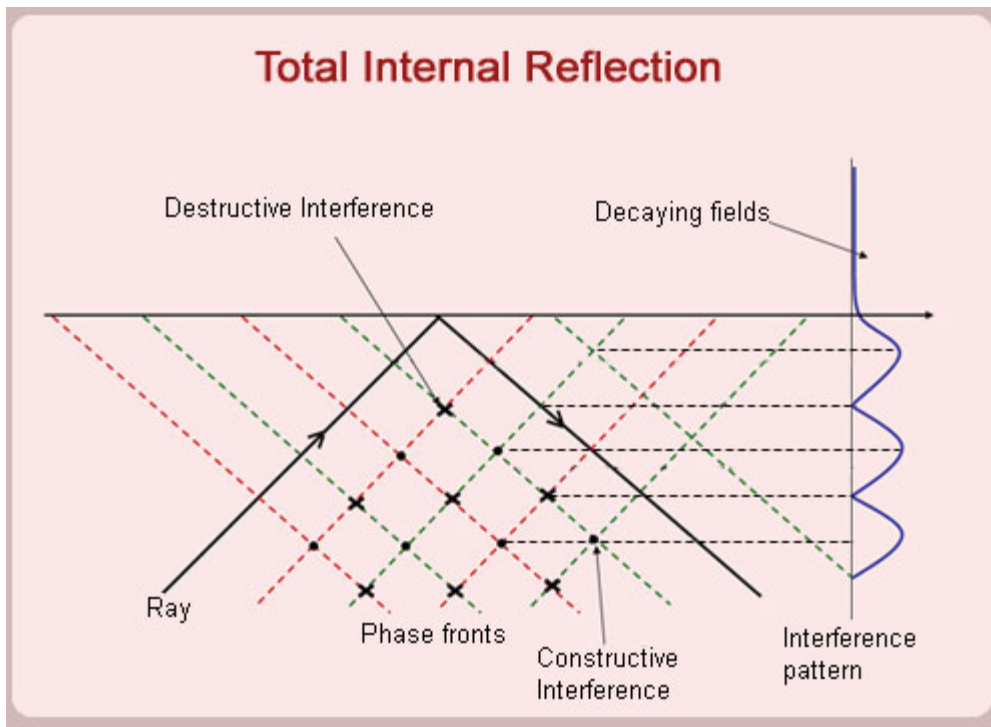Equiprotable rays in a core

Figure (2)

Figure (3): Wave Interference

a) We have seen from the ray model that the rays can be launched at discrete angles inside an optical fiber. For a

particular launching angle all the rays which lie on the surface of a cone are equip-probable rays. Basically the ring of rays is simultaneously launched inside the optical fiber (as shown in the figure 2). Each ray has a wave front associated with it. The wave fronts corresponding to the rays will interfere. Somewhere the interference is destructive and somewhere it is constructive. So when the wave fronts move inside the core of the fiber they exhibit field distributions which have maxima and minima (see fig.3).

(b) When total internal reflection takes place, the field must decay away from the core cladding boundary. If the field does
not decay, then the energy is not guided along the fiber axis and the energy is lost. Here, since we are interested in the guided fields, we accept only those field distributions which decay away from the core-cladding interface.
So the physical understanding of the problem suggests that a solution which has an oscillatory behavior inside the core and decaying behavior inside the cladding is the appropriate choice. Any other solution is not acceptable because it is not consistent with the physical understanding of the modal propagation.

10. Let us now look at the plot of the Bessel functions for various possibilities of q (argument). There are three different types

of Bessel functions depending upon the nature of q.

(a) If q **is real** then the solutions are

$J_v(qr)$ **Bessel functions**

$N_v(qr)$ **Neumann functions**

The quantity ν is called the **order** of the function and (qr) is called the **argument** of the function. Plots of the two functions as a function of their arguments are shown in the Figure 4-5.
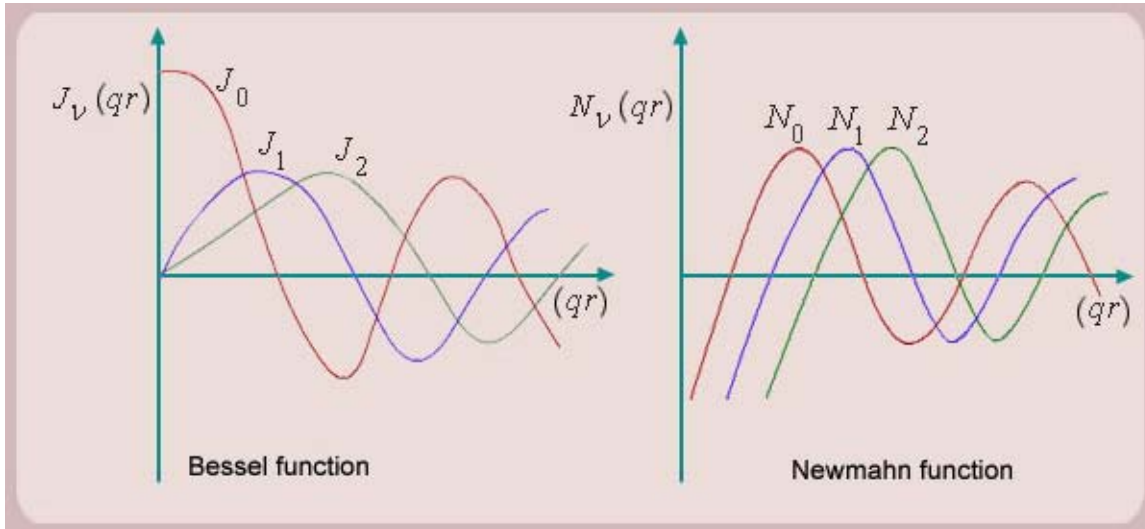

Figure (4) Figure (5)

We can see from figure 4 that except J_0, all the other Bessel functions go to zero as the argument goes to zero. Only J_0 approaches 1 as its argument approaches zero. All Bessel functions have oscillatory behavior and their amplitude slowly decreases as the argument increases.

Figure 5 shows the behavior of the Neumann function as a function of its argument, (qr).

The important thing to note is, the Bessel functions J_ν are finite for all values of the argument, whereas the Neumann functions are finite for all values of argument except zero. When the argument tends to zero, the Neumann functions tend to $-\infty$.

(b) If q **is imaginary,** we get solutions of the Bessel's equation as

$I_\nu(qr/j)$

$K_\nu(qr/j)$

These are called the **Modified Bessel functions** of first and second kind respectively.

Note: Since q is imaginary, (qr/j) is a real quantity. So the argument of the modified Bessel functions is real.

Modified Bessel Function First kind   Modified Bessel Function Second kind
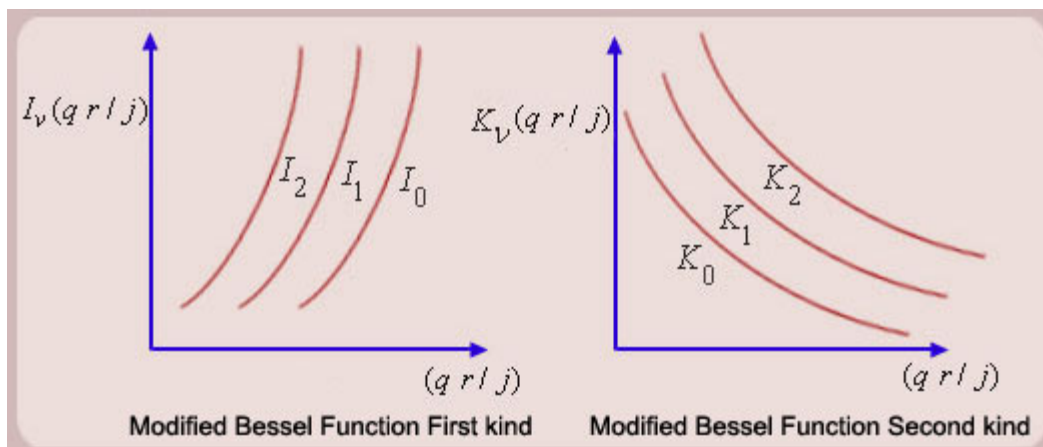
Figure (6) Figure(7)

The modified Bessel functions are shown in the figures 6 and 7. The I_ν functions are monotonically increasing functions of (qr/j), and K_ν functions are monotonically decreasing functions of (qr/j).

(c) If q **is complex**

Then the solutions are

$H_\nu^{(1)}(qr)$ Hankel function of $1st$ kind

$H_\nu^{(2)}(qr)$ Hankel function of $2nd$ kind

In our analysis, q will either be real or imaginary. Therefore we have to deal with Bessel, Neumann and Modified Bessel functions only. The Hankel functions are needed for analyzing propagation in a lossy medium.

(d) Now $q^2 = \omega^2 \mu\varepsilon - \beta^2$, where β is the propagation constant of the wave along the z direction. If we assume the

situation is lossless i.e. when the wave travels in the z direction, its amplitude does not change as a function of z, then β should be a real quantity.

If β becomes imaginary, the function $e^{j\beta z}$ becomes an exponentially decaying function, and there is no wave propagation.

(e) For wave propagation inside an optical fiber we assume that the material is lossless.

Then the dielectric constant ε is a real quantity. This makes

$\omega^2 \mu\varepsilon$ a real quantity.

Also for a propagating mode β^2 is a real quantity.

Hence, $q^2 = \omega^2 \mu\varepsilon - \beta^2$ is also a real quantity albeit it can be positive or negative. In other words, q can be real or imaginary depending upon whether $\omega^2 \mu\varepsilon$ is greater or lesser than β^2. Hence for a lossless situation, solution cannot be given by the Hankel functions.

For a lossless case, we have a solution which is a linear combination of either Bessel and Neumann functions or modified Bessel functions of first and second kind.

(f) As far as guided wave propagation is concerned, the fields should have oscillatory behavior inside the core, and in

cladding the field must decay monotonically. Therefore it is obvious that inside the core the Modified Bessel functions is not the proper solution. Only Bessel function or Neumann function could be solutions inside the core.

Let us take dielectric constant $\varepsilon_{r1}(\equiv n_1^2)$ for the core. Then we have

$$q^2 = \omega^2 \mu \varepsilon_1 - \beta^2 = \omega^2 \mu \varepsilon_0 \varepsilon_{r1} - \beta^2 = \omega^2 \mu \varepsilon_0 n_1^2 - \beta^2$$

For oscillatory type of solution inside the core, q^2 is positive. Therefore for a guided mode we must have

$\omega^2 \mu \varepsilon_1 > \beta^2$ i.e., $\beta < \omega \sqrt{\mu \varepsilon_0} \, n_1 = \beta_0 n_1$

Where β_0 is the wave number in the free space and $\beta_0 n_1 (\equiv \beta_1)$ is nothing but the wave number in an unbound medium of refractive index n_1.

(g) Let us denote q by u inside the core, giving
$$q^2 \equiv u^2 = \beta_1^2 - \beta^2$$

(h) Let us now re-look at the two functions, Bessel functions (fig.4) and Neumann functions (fig.5), and make following observations.

Bessel function: The functions $J_\nu(ur)$ are finite for all values of r.

Neumann Function: The functions $N_\nu(ur)$ start from $-\infty$ at $r = 0$ and have finite value for all other values of r.

For the core $r = 0$ represents the axis of fiber. Therefore if Neumann function is chosen as a solution, the field strength would be ∞ at the axis of the fiber which is inconsistent with the physical conditions. The fields must be finite all over the cross section of the core. So the Neumann functions cannot be the solution if $r = 0$ point is included in the region under consideration.

Therefore we conclude that only $J_\nu(ur)$ is the appropriate solution for the modal fields inside the core of an optical fiber.

11. Field distribution in the cladding is of monotonically decaying nature. We therefore must have q imaginary in the

cladding. We hence should have

$\omega^2 \mu \varepsilon_2 = \omega^2 \mu \varepsilon_0 n_2^2 = \beta_2^2 < \beta^2$ where n_2 is the refractive index of the cladding.
Since $q^2 = \omega^2 \mu \varepsilon_2 - \beta^2$ is negative, let us define a real quantity $w (\equiv q/j)$ such that
$$w^2 = q^2 - \omega^2 \mu \varepsilon_2.$$

Let us now look at the modified Bessel's functions, as shown in figures 6 & 7. For modified Bessel's functions of the $1st$ kind, as r increases, that is, as we move away from the axis of the fiber the field monotonically increases and when $r \to \infty$ field goes to infinity. Since the energy source is inside the core, the fields cannot grow indefinitely away from the core. The only acceptable situation is that the field decays away from the core i.e., for larger values of r. This behavior is correctly given by the Modified Bessel function of second kind, $K_\nu(wr)$.

So we conclude that the modified Bessel function of 1 st kind $I_\nu(wr)$ is not appropriate solution in the cladding. The correct solution would be only Modified Bessel function of 2 nd kind, $K_\nu(wr)$.

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(a) In all then, the fields inside the core are given by $J_\nu(ur)$ and in the cladding are given by $K_\nu(wr)$.

(b) For a guided mode, the propagation constant lies between two limits β_1 and β_2.

If $\beta_1 \le \beta \le \beta_2$ then a field distribution is generated which will has an oscillatory behavior in the core and a decaying behavior in the cladding. The energy then is propagated along fiber without any loss.

(c) **Field distribution:** From the solution of the wave equation we get the longitudinal fields inside the core and the

cladding as

Inside Core $(r < a)$

Electric field:

$E_{z1} = A J_\nu(ur) e^{j\nu\phi - j\beta z + j\omega t}$ **(11a)**

Magnetic field:

$H_{z1} = B J_\nu(ur) e^{j\nu\phi - j\beta z + j\omega t}$ **(11b)**

In Cladding $(r > a)$

Electric field:

$E_{z2} = C K_\nu(wr) e^{j\nu\phi - j\beta z + j\omega t}$ **(11c)**

Magnetic field:

$H_{z2} = D K_\nu(wr) e^{j\nu\phi - j\beta z + j\omega t}$ **(11d)**

Where A, B, C, D are arbitrary constants which are to be evaluated from the boundary conditions.

(d) Once we get the longitudinal components of the electric and magnetic fields, we can find the transverse field

components inside the core and the cladding way the relations given above.

Applying the boundary conditions i.e., the tangential components of electric field and the tangential components of magnetic field are continuous along the core cladding boundary, we get what is called the characteristic equation of the mode.

(e) The tangential components at the core cladding boundary are the ϕ and the z components.

The boundary conditions are then given as:

At $r = a$,

1. $E_{\phi 1} = E_{\phi 2}$

2.

$E_{z1} = E_{z2}$

3. $H_{\phi 1} = H_{\phi 2}$

4. $H_{z1} = H_{z2}$

The boundary conditions give four equations in terms of arbitrary constants, A, B, C, D and the modal phase constant, β.

(f) We find the equation for the propagation constant, β of the wave, by eliminating the arbitrary constants. Elimination of
the arbitrary constants gives the characteristic equation of mode inside an optical fiber as

Characteristic Equation (Eigen Value Equation)

$$\left\{ \frac{J_v'(ua)}{uJ_v(ua)} + \frac{K_v'(wa)}{wK_v(wa)} \right\} \left\{ \beta_1^2 \frac{J_v'(ua)}{uJ_v(ua)} + \beta_2^2 \frac{K_v'(wa)}{wK_v(wa)} \right\} = \frac{\beta^2 v}{a} \left\{ \frac{1}{u^2} + \frac{1}{w^2} \right\}^2 \qquad \text{(12a)}$$

where denotes derivative with respect to the argument i.e.,

$J'_v(x) = \partial J_v(x)/\partial x \ \text{and} \ K'_v(x) = \partial K_v(x)/\partial x$.

The characteristic equation contains three unknowns namely u, w, β. However, there are two more equations

$$u^2 = \omega^2 \mu \varepsilon_1 - \beta^2 \qquad \text{(12b)}$$

$$w^2 = \beta^2 - \omega^2 \mu \varepsilon_2 \qquad \text{(12c)}$$

So using the equations (12a), (12b) & (12c) we can find the modal propagation constant β.

(h) If β is real, the mode propagates and if β is imaginary the mode is evanescent.

(i) We have to use numerical techniques to solve the characteristic equation for a given value of v. We get multiple
solutions to the problem and each solution gives one mode for a given value of v. So depending upon the fiber radius, different number of modes propagates inside an optical fiber.

3. Modes of optical fiber

1. A mode can be characterized by two parameters, v and the solution number.

2 (a) If we take $H_z = 0$, all field components are expressed in terms of E_z and whatever fields we get, they do not have any

magnetic field component in the direction of propagation. We call this mode the **Transverse Magnetic mode (TM mode).**

Similarly if $E_z = 0$, the mode is called the **Transverse Electric mode (TE mode).**

(b) If both the longitudinal components of the fields (E_z and H_z) are non-zero then we call the mode the **Hybrid mode.**
This mode is a combination of TE and TM modes.

(c) So inside an optical fiber we have three types of modes
1. **TE modes**
2. **TM modes**
3. **Hybrid modes**

(d) For a hybrid mode, if we calculate the contribution by E_z and H_z to the transverse fields, one of them i.e. E_z or H_z would

dominate. Depending upon which of them contributes more, we can sub-classify the Hybrid modes.
If E_z Dominates $\Rightarrow$ **EH mode**
If H_z Dominates $\Rightarrow$ **HE mode**

4. Each of the above three modes are characterized by two indices, ν and m (solution number). The mode are therefore
designated as $TE_{\nu m}, TM_{\nu m}$ & $EH_{\nu m}, HE_{\nu m}$

5. What do these two indices physically mean?

In a rectangular wave guide, they represent the number of half cycle variations and the number of zero crossings in the x direction and in y directions. The same thing is applicable in this case also.

(a) If $\nu = 0$, the function $e^{i\nu\phi}$ is constant and the field distribution is circularly symmetric.

(b) For $\nu = 1$ we get the one cycle variation in ϕ direction, if $\nu = 2$ we get two cycle variation in the ϕ direction, and so on.

(c) The first index of the mode therefore gives the zero crossings in the azimuth of the cross section of the optical fiber.

The second index tells us how many zero crossings the field distribution has in the radial direction. So if we fix all other parameters and just move radially outwards, how many zero crossings the field variation would see is essentially given by the second index. If we don't have any zero crossing then we have $m = 1$, if we have one zero crossing then $m = 2$, and so on.

6. For $\nu = 0$ the characteristic equation (12a) becomes

$$\left\{ \frac{J_0'(ua)}{uJ_0(ua)} + \frac{K_0'(wa)}{wK_0(wa)} \right\} \left\{ \beta_1^2 \frac{J_0'(ua)}{uJ_0(ua)} + \beta_2^2 \frac{K_0'(wa)}{wK_0(wa)} \right\} = 0 \quad (13a)$$

So either of the brackets could be equal to zero.

(a) If we take first bracket equal to zero, the equation gives the characteristic equation of the transverse electric mode.
Similarly, the second bracket equal to zero, gives the characteristic equation of the transverse magnetic mode.

Since   represents transverse electric and transverse magnetic modes, the   and

(b) $\nu = 0$ modes have field distributions which are essentially circularly symmetric.

If $\nu > 0$ then we always get a field distribution which is hybrid. So the transverse electric and transverse magnetic fields have only radial variation, and they do not have any variation in the ϕ direction.

(c) If we take the first bracket equal to zero, we get

$$\left\{ \frac{J_0{'}(ua)}{uJ_0(ua)} + \frac{K_0{'}(wa)}{wK_0(wa)} \right\} = 0 \underline{\quad\quad} \text{(2-13b)}$$

This is the **characteristic equation for TE_{0m} mode.**

We get multiple solutions for this equation because J_0 function is an oscillatory Function.

Using recurrence relation for the Bessel function, we have $J_0{'}(x) = -J_1(x)$ and $K_0{'}(x) = -K_1(x)$.

The characteristic equation for the TE mode therefore becomes

$$\frac{J_1(ua)}{uJ_0(ua)} + \frac{K_1(wa)}{wK_0(wa)} = 0$$

Similarly the characteristic equation for the TM mode is

$$\beta_1^2 \frac{J_1(ua)}{u\,J_0(ua)} + \beta_2^2 \frac{K_1(wa)}{w\,K_0(wa)} = 0$$
.

Important: Inside an optical fiber a finite number of modes propagates. Also a particular mode propagates if the frequency of light is greater than certain value. For a given optical fiber it is then important to find the frequency range over which a particular mode propagates. This aspect is discussed in the next module.

Recap

In this lecture you have learnt the following

- **Analysis of cylindrical waveguides**

- Characteristic Equation (Eigen Value Equation)

- **Modes of optical fiber**

Lecture 3.2 : Wave Model - II

Objectives
In this lecture you will learn the following

- **Cut off frequency of a mode**

- **V Number of Optical Fiber**

- E-field distributions for various modes

- **Cut off conditions for various modes**

- **Objective of Modal Analysis**

- **Weakly Guiding Approximation**

- **Linearly polarized modes**

MODAL PROPAGATION INSIDE AN OPTICAL FIBER (WAVE MODEL-II)

1. **CUT OFF FREQUENCY OF A MODE**
   As seen earlier, β has to be real for a propagating mode. The frequency range over which β remains real therefore is important information. It can be shown that for β to be real the frequency of the wave has to be greater than certain value, called the cut-off frequency.

   1. Cutoff frequency is defined as the frequency at which the mode does not remain purely guided. That is, when a guided mode is converted into a radiation mode.

   2. The cut-off is defined by $w = 0$ (and not $\beta = 0$ as is usually done for the metallic waveguides) where, $w^2 = \beta^2 - \omega^2 \mu \varepsilon_2$, and $\omega \sqrt{\mu \varepsilon_2}$ is propagation constant in cladding (β_2).

      If w is real we get the guided mode

and if w is imaginary we get radiating mode

3. At cut off frequency

   since $w \to 0$,

   $$\beta \to \beta_2$$

   which means that the propagation constant of the wave approaches the propagation constant of a uniform plane wave in the medium having dielectric constant ε_2.

2 . V Number of Optical Fiber

1. The V-number is one of the important characteristic parameters of a step index optical fiber. V-number of an optical fiber

   is defined as

   $$V = \frac{2\pi}{\lambda} a \left(n^2_1 - n^2_2 \right)^{\frac{1}{2}} \text{ (1)}$$

   $$= \left(\beta_1^2 - \beta_2^2 \right)^{\frac{1}{2}} a$$

   Now since

   $$u^2 = \omega^2 \mu \varepsilon_2 - \beta^2$$

   and $w^2 = -\omega^2 \mu \varepsilon_2 + \beta^2$

   we have $u^2 + w^2 = \omega^2 \mu \varepsilon_1 - \omega^2 \mu \varepsilon_2$

   $$= \beta_1^2 - \beta_2^2$$

   If we multiply both sides by a^2 we get

   $$a^2 \left(u^2 + w^2 \right) = a^2 (\beta_1^2 - \beta_2^2) = V^2 \text{ (2)}$$
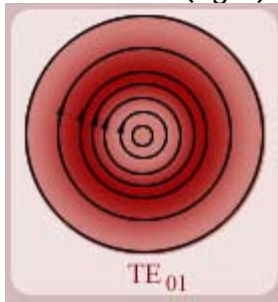
   At the cutoff when $w = 0$, we have $ua = V$ number.

   The V-number provides information about the modes on a step index fiber. As can be seen, the V-number is proportional to the numerical aperture, and radius a of the core, and is inversely proportional to wavelength λ.

   E-field distributions for various modes:

HE_{11} Mode (fig 1)

TE_{01} Mode (fig 2)

TM_{01} Mode (fig 3)

HE_{21} Mode (fig 4)

It can be noted from the table that for the HE_{11} mode at cut-off $ua = V = t_{11} = 0$. That is, HE_{11} mode does not have any cut off.

HE_{11} mode is a special mode as it always propagates. This mode does not have a cutoff.

From table, we note that, HE_{21} mode shows cutoff when $ua = V = t_{2-2, m} = t_{0m}$. The cut off for

TE_{0m} and TM_{0m} modes is given by $ua = V = t_{0m}$. Hence we note that the cutoff frequencies of TE_{0m}, TM_{0m} and HE_{21} is the same. Since $t_{01} = 2.4$ (first root of J_0 Bessel function), below $V = 2.4$ only HE_{11} mode propagates.

Important: (i) The lowest order mode on the optical fiber is the HE_{11} mode.

(ii) The fiber remains single mode if its V-number is less than 2.4.

For TE_{01}, TM_{01} modes we have

$ua = 2.4$ since it is the maximum value of ua for guided wave propagation.

For single mode propagation,

$V \le 2.4$

$$\Rightarrow \frac{2\pi}{\lambda} a \left(n_1^2 - n_2^2 \right)^{\frac{1}{2}} \le 2.4 ,$$

For a typical fiber the numerical aperture $= \left(n_1^2 - n_2^2 \right)^{\frac{1}{2}} = 0.1$

This gives

$$\Rightarrow 2\pi \left(\frac{a}{\lambda} \right) 0.1 = 2.4$$

$$\Rightarrow \frac{a}{\lambda} \approx \frac{2.4}{0.6}$$

$$\Rightarrow \frac{a}{\lambda} \approx 4 \, (3)$$

So the effective radius is very small for single mode optical fiber which is why the normal light source will not become the source. Only LASERS type source is essential to launch light inside single mode optical fiber and the cross section becomes so small that it accepts only one ray corresponding to HE_{11} mode.

3. **CUT OFF CONDITIONS FOR VARIOUS MODES**

The table gives the value of ua at cut off for different modes

MODES	CUTOFF
TE_{0m}, TM_{0m}	t_{0m}
HE_{1m}	$t_{1,m-1}$

EH_{vm} $\qquad\qquad\qquad\qquad\qquad t_{vm}$

HE_{vm} $\left(\dfrac{n_1^2 - n_2^2}{n_1^2} <<1 \right)$ $t_{v-2,m}$

where t_{pq} is the qth root of the J_p Bessel function.

4. Objective of Modal Analysis

a. Primarily we are interested in velocity of different modes since this information helps in obtaining the amount of the
pulse spread, i.e., dispersion.

b. The phase and group velocities of a mode are given as

Phase velocity $= \dfrac{\omega}{\beta}$

Group velocity $= \dfrac{d\omega}{d\beta}$

c. Variation of β as a function of frequency is the primary outcome of the model analysis.

From equation (1) we note that V number $= \dfrac{2\pi}{\lambda} a \left(n_1^2 - n_2^2 \right)^{\frac{1}{2}}$

$= \dfrac{\omega}{c} a (NA)$ (4)

For a given fiber, the radius (a) is fixed and the numerical aperture (NA) is also fixed. We therefore get

$\Rightarrow V \propto \omega$ (V number is proportional to the frequency of the wave).

Since V is proportional to ω, instead of writing variation of β in terms of ω, we can obtain variation of β as a function of V-number of an optical fiber. The V-number hence is called the **Normalized frequency.**

d. For a guided mode we have

$\beta_2 \leq \beta \leq \beta_1$

The value of β can vary over a wide range depending upon the fiber refractive indices and

the wavelength.

Let us therefore define the Normalized propagation constant b as

$$b = \frac{\beta^2 - \beta_2^2}{\beta_1^2 - \beta_2^2} \quad (5)$$

b always lies between 0 and 1.

We can see from the above equation that if mode is very close to cutoff, then $\beta \to \beta_2$ and $b \to 0$.

On the other hand when a mode is very far from cutoff then $\beta \to \beta_1$ and $b \to 1$.

A plot of b v/s V is called the $b-V$ diagram (fig.5). The $b-V$ diagram is the characteristic diagram for propagation of modes in a step index optical fiber.
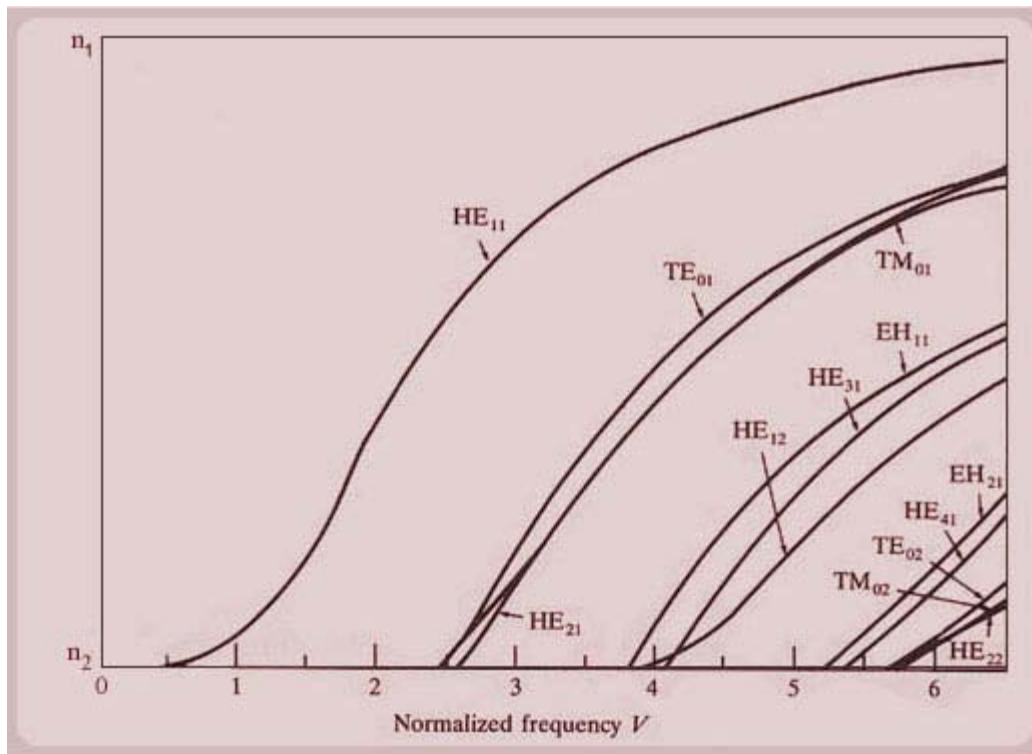
$b-V$ Diagram (fig 5)

Explanation of b-V diagram :

(a) The $b-V$ plot for every mode is a monotonically increasing function of the V-number. Every plot starts at $b = 0$ and asymptotically saturates to $b = 1$. That suggests that the modal fields get more and more confined with increasing $V-number$.

(b) The HE_{11} mode propagates for any value of $V > 0$.

(c) TE_{01}, TM_{01} and HE_{21} modes propagate for $V > 2.4$.

(d) All the mode which have cut-off V-number less than the V-number of the fiber, propagate inside the fiber.

5. Weakly Guiding Approximation

For a practical fiber, the difference is the refractive indices of the core and the cladding is very small. This justifies certain approximation in the modal analysis. This approximation is called the 'weakly guiding approximation'. The characteristic equations then can be simplified in this situation. The approximate but simplified analysis can provide better insight into the modal characteristics of an optical fiber. In weakly guiding situation there is substantial spread of fields in the cladding. The optical energy is not tightly confined to the core and is weakly guided.

a. For weakly guiding fibers, all the modes which have same cutt-off V-numbers degenerate, that is, their $b - V$ curves

almost merge. For example, for a weakly guiding fiber, the three modes TE_{01}, TM_{01} and HE_{21} would degenerate. These modes then have same phase velocity. The modal fields corresponding to the three modes the travel together with same phase change. Consequently the three modal patterns are not seen distinctly but we get a super imposed field distribution.

b. The superimposed field distributions are linearly polarized in nature. That is, the field orientation is same every where

in the cross-sectional plane. Since the fields are linearly polarized, the modes are designated as the Linearly polarized (LP) modes.

As an example we can see that (see Fig. 6)

c. If TE_{01} mode superimposes on to HE_{21} mode, the field distribution is horizontally polarized.

d. If TM_{01} mode superimpose on to HE_{21} mode the field distribution is in vertically polarized.
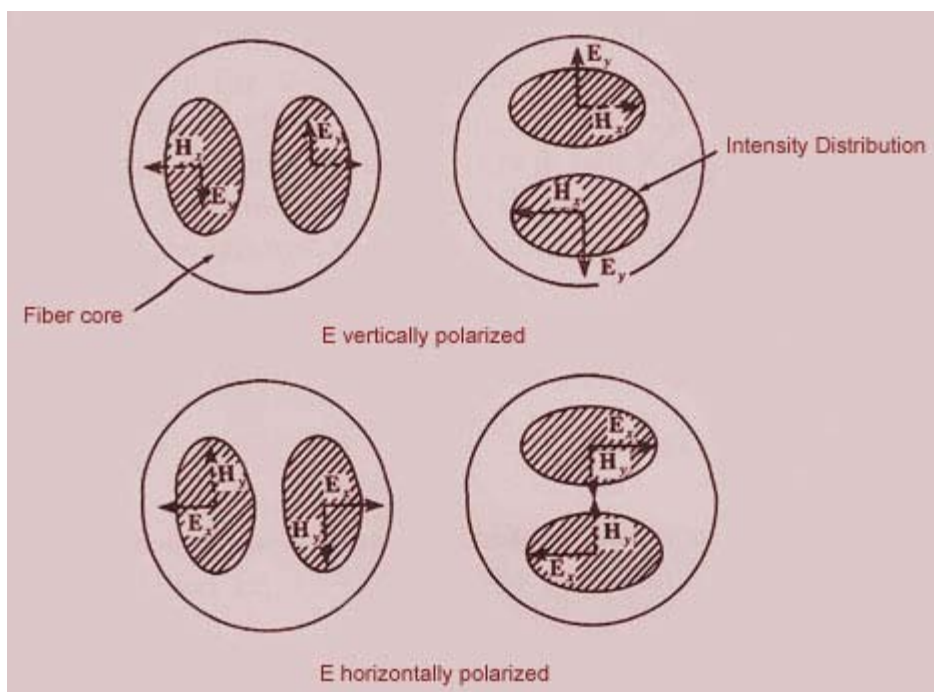

Fig-6

6. LINEARLY POLARIZED MODES

1. LP Modes are not the fundamental modes like the TE, TM and hybrid modes of an optical fiber. However, since we are

unable to distinguish the fundamental modes inside a weakly guiding fiber, we get a field distribution which looks like a linearly polarized mode. The LP modes also have two indices. Consequently the LP modes are designated as LP_{vm} modes. The mapping of fundamental modes to LP modes is given in the following:

$$HE_{11} \rightarrow LP_{01} \quad \text{(6a)}$$

$$TE_{0m} + TM_{0m} + HE_{2m} \rightarrow LP_{1m} \quad \text{(6b)}$$

$$HE_{v+1,m} + EH_{v-1,m} \rightarrow LP_{vm}, \quad v \geq 2 \quad \text{(6c)}$$

2. The total number of modes (M) propagating inside a fiber is approximately given as

$$M = \frac{V^2}{2}$$

The relation is approximate and is useful for large V-numbers. The approximations are reasonably accurate for $v \gtrsim 5$.

Conclusion :

The light propagates in the form of modes inside an optical fiber. Each mode has distinct electric and magnetic field patterns. On a given fiber, a finite number of modes propagates at a given wavelength. Intrinsically the model fields could be TE, TM or Hybrid, however for weakly guiding fibers modal fields become linearly polarized. The $b - V$ diagram is the universal plot for a step index fiber. The diagram provides information regarding the cut-off frequencies of the modes, and the number of propagating mode, phase and group velocities of a mode. As will be seen in the next module, the modal analysis provides the base for estimating dispersion on the optical fiber.

Recap

In this lecture you have learnt the following

- **Cut off frequency of a mode**

- **V Number of Optical Fiber**

- E-field distributions for various modes

- **Cut off conditions for various modes**

- **Objective of Modal Analysis**

- **Weakly Guiding Approximation**

- **Linearly polarized modes**

Congratulations, you have finished Module 3. To view the next lecture select it from the left hand side menu of the page