UNIT II

Fourier series & transform Representation of Continuous Time Signals

Learning Objectives:

- To introduce the concept of various Fourier series
- To introduce the concept of convergence of Fourier series.
- To introduce the concept of representation of a periodic function by Fourier series
- To introduce the concepts of sampling of continuous time signlas.
- To introduce the concept of Fourier transform.

Learning Outcomes:

Students will be able to

- Apply symmetric conditions
- Represent the periodic function by Fourier series
- Obtain the relation between trigonometric and exponential Fourier series
- Perform transformations on signals

Syllabus:

Trigonometric and exponential Fourier series, relationship between trigonometric and exponential Fourier series, representation of a periodic function by the Fourier series over the entire interval, convergence of Fourier series, alternative form of trigonometric series, **symmetry conditions:** Even, Odd and Half-wave symmetry. **Properties of Fourier series:** linearity, time scaling, time shifting, time reversal, differentiation, integration, modulation, convolution and Parseval's theorem. Complex Fourier transforms.

Representation of an arbitrary function over the entire interval: Fourier transform, Existence of Fourier transform, Fourier transform of some useful functions, Fourier transform of periodic function, Properties of Fourier transform, Energy Density spectrum, Parseval's Theorem.

Introduction

Many of the phenomena studied in engineering and science are **periodic** in nature eg. The current and voltage in an alternating current circuit. These periodic functions can be analysed into their constituent components (fundamentals and harmonics) by a process called **Fourier analysis.**

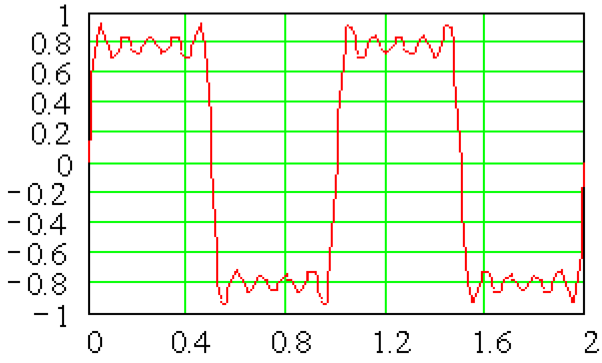
Fourier series was developed by **Joseph Fourier**

Definition

. Fourier series is a mathematical tool which decomposes any periodic function or periodic signal into sum of set of sine and cosine terms.

OR

The representation of signals over a certain time interval interns of the linear composition of orthogonal functions is called **Fourier series**. It is sometimes called the **Harmonic analysis.**



There are 3 important types of Fourier series. They are

1. Trigonometric form
2. Cosine form
3. Exponential form

Trigonometric Fourier series

The representation of signals over a certain time interval interns of the linear composition of trigonometric functions is called **Trigonometric Fourier series.**

The trigonometric Fourier series representation of any periodic signal x (t) is defined as

$$x(t) = a_o + \sum_{n=1}^{\infty}[a_n cosnw_o t + b_n sinnw_o t] \qquad (1)$$

Wherea_n, b_n are called constants and a_o is called dc component and is given by

$$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)cosnw_o t\, dt$$

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)sinnw_o t\, dt$$

Exponential Fourier series

The representation of signals over a certain time interval interns of the linear composition of exponential functions is called **Exponential Fourier series.** The complex exponential form is more general and usually more convenient & more compact when compared to Trigonometric Fourier series.

The exponential Fourier series of signal x (t) is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o t}$$

OR

$$x(t) = c_0 + \sum_{n=-\infty}^{-1} c_n e^{jnw_o t} + \sum_{n=1}^{\infty} c_n e^{jnw_o t} \qquad (2)$$

Relationship between trigonometric and exponential Fourier series

The complex exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o t}$$

$$= c_0 + \sum_{n=-\infty}^{-1} c_n e^{jnw_o t} + \sum_{n=1}^{\infty} c_n e^{jnw_o t}$$

$$= c_0 + \sum_{n=1}^{\infty} (c_{-n} e^{-jnw_o t} + c_n e^{jnw_o t})$$

$$= c_0 + \sum_{n=1}^{\infty} c_{-n} (cosnw_o t - jsinnw_o t) + c_n(cosnw_o t + jsinnw_o t))$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} (c_{-n} + c_n)\, cosnw_o t + j(c_n - c_{-n}) sinnw_o t) \qquad (3)$$

Compare eq(1) and eq(3)

$$a_0 = c_0$$

$$a_n = c_n + c_{-n}$$

$$b_n = j(c_n - c_{-n})$$

Similarly

$$c_0 = a_0$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Representation of a Periodic Function by the Fourier Series Over the Entire Interval

So far we have considered the Fourier series representation of an arbitrary function over finite interval between $t_o and\ (t_o + T)$. The function and its Fourier series may not be equal outside this interval. However, a periodic function which repeats after every T seconds is expected to have the same Fourier series for the entire interval (-∞, ∞). A periodic function has an identical Fourier series for the entire interval (-∞, ∞) as for the interval $(t_o + T)$ since same function is repeated for every T seconds. Thus for a periodic function x (t) we have

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jnw_o t}\ ,\ (-\infty‹t‹\infty)$$

$$F_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t) e^{-jnw_o t} dt$$

Dirichlets Conditions (Convergence of Fourier Series)

For the Fourier series to exist for a periodic signal it must satisfy certain conditions and they are

1. Function x(t) must be a single valued function
2. Function x(t) has only finite number of maxima & minima.
3. Function x(t) has finite number of discontinuities.
4. Function x(t) is absolutely integral over one period i,e $\int_0^T |x(t)| dt < \infty$

Alternative form of Fourier series (Cosine Fourier series)

The trigonometric Fourier series of x(t) contains "sine" and "cosine" terms of the same frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n cosnw_0 t + b_n sinnw_0 t]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left[\frac{a_n}{\sqrt{a_n^2+b_n^2}}cosnw_0 t + \frac{b_n}{\sqrt{a_n^2+b_n^2}}sinnw_0 t\right]$$

Substituting the values $a_0 = A_0$, $A_n = \sqrt{a_n^2+b_n^2}$, $\theta_n = 1\tan^{-1}\frac{b_n}{a_n}$, $\frac{a_n}{\sqrt{a_n^2+b_n^2}} = cos\theta_n$ and $\frac{b_n}{\sqrt{a_n^2+b_n^2}} = -sin\theta_n$ in the above equation of x(t) we get

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n\ [cos\theta_n cosnw_0 t - sin\theta_n sinnw_0 t]$$

$$x(t) = A_0 + \textstyle\sum_{n=1}^{\infty} A_n\ [\cos(nw_0 t + \theta_n]$$

The term A_0 is called the dc component. The term A_n represents the amplitude coefficients (or) harmonic amplitudes (or) spectral amplitudes of the Fourier series and θ_n represents the phase coefficients (or) phase angles of Fourier series. The cosine form is also called as "Harmonic Form" (or) "Polar Form" Fourier series.

Symmetric conditions or Wave symmetry

There are 4 types of symmetry of function x(t) can have

1. Even symmetry
2. Odd symmetry
3. Half wave symmetry

Even symmetry

A function x(t) is said to be even symmetry if

$$x(-t) = x(t)$$

These types of functions always symmetrical w.r.t the vertical axis

$$x(t) = x_e(t) + x_0(t)$$

If x(t) is even function then, $x_0(t) = 0$ and $x_e(t) = x(t)$. When even symmetry exists the trigonometric Fourier series coefficients are

$$a_o = \frac{2}{T}\int_0^{T/2} x(t)dt$$

$$a_n = \tfrac{4}{T}\textstyle\int_0^{T/2} x(t)cosnw_o t\, dt$$

$$b_n = 0$$

Odd symmetry

A function x(t) is said to be odd symmetry if

$$x(-t) = -x(t)$$

These types of functions always antisymmetrical w.r.t the vertical axis

$$x(t) = x_e(t) + x_0(t)$$

If x(t) is odd function then, $x_e(t) = 0$ and $x(t) = x_0(t)$. When odd symmetry exists the trigonometric Fourier series coefficients are

$$a_o = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t) sinnw_o t\, dt$$

Half wave symmetry

A periodic signal x(t) which satisfies the condition

$x(t) = -x\left(t \pm \frac{T}{2}\right)$ is said to be half wave symmetry.

The Fourier series expansion of half wave symmetry signal contains odd harmonics only (When **"n"** is odd).

$$a_n = \frac{4}{T}\int_0^{T/2} x(t) cosnw_o t\, dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t) sinnw_o t\, dt$$

When "n" is even

$$a_n = 0$$

$$b_n = 0$$

Properties of Fourier Series

Let us consider $x_1(t) and\ x_2(t)$ are two periodic signals with period 'T' and with Fourier series coefficients $C_n \& D_n$.

1. **Linearity Property**

The linearity property states that, if

$$FS\{x_1(t)\} = C_n\ and\ FS\{x_2(t)\} = D_n$$

Then $FS\ \{A\ x_1(t) + Bx_2(t)\} = AC_n + B\ D_n$

Proof:

From the definition of Fourier series, we have

$$FS\{A\ x_1(t) + Bx_2(t)\} = \frac{1}{T}\int_{t_o}^{t_o+T} [A\ x_1(t) + Bx_2(t)] e^{-jnw_0 t} dt$$

$$FS\{A\ x_1(t) + Bx_2(t)\} = A\frac{1}{T}\int_{t_o}^{t_o+T} [\ x_1(t)] e^{-jnw_0 t} dt + B\frac{1}{T}\int_{t_o}^{t_o+T} [x_2(t)] e^{-jnw_0 t} dt$$

$$FS\{A\ x_1(t) + Bx_2(t)\} = AC_n + B\ D_n$$

2. **Time Reversal**

The time reversal property states that, if

$$Fs\{x(t)\} = C_n$$

Then $Fs\{x(-t)\} = C_{-n}$

Proof:

From the dentition of Fourier series, we have

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o t}$$

$$x(-t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o(-t)}$$

Substitute “n = -p” in the right hand side, we get

$$x(-t) = \sum_{p=\infty}^{-\infty} c_{-p} e^{j(-p)w_o(-t)} = \sum_{p=-\infty}^{\infty} c_{-p} e^{jpw_o t}$$

Substitute “p = n”, we get

$$x(-t) = \sum_{n=-\infty}^{\infty} c_{-n} e^{jnw_o t} = FS^{-1}\{c_{-n}\}$$

Therefore $Fs\{x(-t)\} = C_{-n}$

3. **<u>Time Shifting</u>**

The time shifting property states that, if

$$Fs\{x(t)\} = C_n$$

Then $FS\ \{x(t - t_0) = e^{-jnw_o t_0} C_n$

Proof:

From the dentition of Fourier series, we have

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o t}$$

$$x(t - t_0) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o(t-t_0)}$$

$$x(t - t_0) = \sum_{n=-\infty}^{\infty} \left[c_n e^{-jnw_o t_0}\right] e^{jnw_o t}$$

$$x(t - t_0) = FS^{-1}[c_n e^{-jnw_o t_0}]$$

Therefore $FS\ \{x(t - t_0) = C_n e^{-jnw_o t_0}$

4. **<u>Time Scaling:</u>**

The time scaling property states that, if

$$Fs\{x(t)\} = C_n$$

Then

$$Fs\{x(at)\} = C_n \; with \; \omega_0 \to a\omega_0$$

Proof:

From the definition of Fourier series, we have

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o t}$$

$$x(at) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o at}$$

$$x(at) = \sum_{n=-\infty}^{\infty} c_n e^{jn(aw_o)t}$$

$$x(at) = FS^{-1}\{c_n\}$$

Where $\omega_0 \to a\omega_0$

Therefore

$$Fs\{x(at)\} = C_n \; with \; fundamental \; frequency \; of \; a\omega_0$$

5. **Convolution Property**

The convolution property states that, if

$$FS\{x_1(t)\} = C_n \; and \; FS\{x_2(t)\} = D_n$$

Then

$$FS\{x_1(t) * x_2(t)\} = C_n D_n$$

6. **Time Differentiation Property**

The time differentiation property states that, if

$$Fs\{x(t)\} = C_n$$

Then $$FS\left\{\frac{dx(t)}{dt}\right\} = jn\omega_0 C_n$$

7. **Time Integration Property**

The time integration property states that, if

$$Fs\{x(t)\} = C_n$$

Then $$FS\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{C_n}{jn\omega_0} \text{ (if } C_0 = 0)$$

8. **Modulation or Multiplication Property**

The modulation property states that, if

$$FS\{x_1(t)\} = C_n \; and \; FS\{x_2(t)\} = D_n$$

Then $$FS\{x_1(t) \times x_2(t)\} = \sum_{l=-\infty}^{\infty} C_l D_{n-l}$$

9. **Parseval's Theorem**

If

$$FS\{x_1(t)\} = C_n \; and \; FS\{x_2(t)\} = D_n$$

Then parseval's theorem states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)x_2^*(t)dt = \sum_{n=-\infty}^{\infty} C_n D_n^*$$

Complex Fourier Transform

The Fourier spectrum of a periodic signal x(t) is a plot of its Fourier coefficients versus frequency (ω). It is in two parts

1. Amplitude Spectrum
2. Phase Spectrum

Amplitude Spectrum

The plot of the amplitude of Fourier coefficients versus frequency (ω) is known as amplitude spectrum.

Phase Spectrum

The plot of the phase of Fourier coefficients versus frequency (ω) is known as phase spectrum.

The two plots together are known as Fourier frequency spectra of x(t). It is also known as frequency domain representation. The Fourier spectrum exists only at discrete frequencies (nw_0) where n = 0, 1, 2 ... hence it is also known as discrete spectrum.

The trigonometric representation of a periodic signal x(t) contains both sine & cosine terms with positive and negative amplitude coefficients but will have no phase components.

The cosine representation of a periodic signal contains only positive amplitude coefficients with phase angleθ_n. Therefore, we can plot amplitude spectra and phase spectra. Since in this representation Fourier series coefficients exist only for positive frequencies. This spectrum is called single-sided spectra.

The exponential representation of a periodic signal x(t) contains amplitude coefficients c_n which are complex. Hence, they can be represented by magnitude and phase. Therefore we can plot two spectra, the magnitude spectrum ($|c_n|\ versus\ \omega$) and phase spectrum ($angle\ c_n\ versus\ \omega$). The spectra can be plotted for both positive & negative frequencies. Hence it is called two-sided spectra.

The magnitude spectrum is symmetrical about the vertical axis passing through the origin and phase spectrum is antisymmetrical about the vertical axis passing through origin. So magnitude spectrum exhibits even symmetry & phase spectrum exhibits odd symmetry.

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I Objective Questions

1. Define Fourier series
2. Explain Dirichlet conditions of fourier series
3. Give the conditions for Even and Odd Symmetry of a periodic signal.
4. Give Conditions for existence of Fourier series.
5. Define Trigonometric Fourier series.
6. The trigonometric Fourier series of an even function of time does not contain [ ]

   (a) Sine term (b) odd harmonic term

   (c) Cosine term (d) DC term
7. If x(t) and y(t) are two periodic signals with the Fourier series coefficients c_n and d_n, then the Fourier coefficients of z(t) = x(t) * y(t) are [ ]

   (a) $\frac{1}{T} c_n d_n^*$ (b) Tc_n d_n (c) c_n d_n (d) $c_n * d_n$
8. The trigonometric Fourier series of a periodic time function have [ ]

   (a) Sine terms (b) cosine terms

   (c) DC term and (a) and (b) (d) both (a) and (b)
9. Give the expressions for C_n and C_{-n} .
10. Define Exponential Fourier series.

II Descriptive Questions

1. Write the any 4 properties of Fourier series?
2. explain the differentiation and integration properties of Fourier series
3. Obtain the relationship between trigonometric and exponential Fourier series.
4. Explain the concept of complex Fourier spectrum.

PART B

I Multiple Choice Questions

1. x(t) be the real-valued signal with fundamental period T and Fourier coefficients c_n then Fourier coefficients of $x_e(t)$ are equal to

(a) $Re\{c_n\}$ (b) c_n (c) $Im\{c_n\}$ (d) $jIm\{c_n\}$

2. For the given periodic function $f(t) = \begin{cases} 2t & for\ 0 \le t \le 2 \\ 4 & for\ 2 \le t \le 6\ (= T) \end{cases}$ .The coefficient $b_n$of the continuous Fourier series associated with the given function f (t) can be computed as

(a) − 75.6800 (b) − 7.5680 (c) − 6.8968 (d) − 0.7468

3. Which of the following is an "even" function of 't' ?

(a) t^2 (b) t^2-4t (c) sin(2t) + 3t (d) t^3 +6

4. A "periodic function" is given by a condition which

(a) Has a period T=2π (b) Satisfies $f(t+T) = f(t)$

(c) Satisfies $f(t+T) = -f(t)$ (d) Has a period T=π

5. For the given periodic function $f(t) = \begin{cases} 2t & for\ 0 \le t \le 2 \\ 4 & for\ 2 \le t \le 6\ (= T) \end{cases}$ the coefficient $a_n$of the continuous Fourier series associated with the given function f (t) can be computed as

(a) -9.2642 (b) -8.125 (c) -0.9119 (d) -0.5116

6. If $Fs\{x(t)\} = C_n$ then FS{x (2t)} is given by

(a) $C_n\ with\ \omega_0 \to 2\omega_0$ (b) $C_n\ with\ \omega_0 \to 1/2\omega_0$

(c) $1/C_n\ with\ \omega_0 \to 2\omega_0$ (d) $C_n\ with\ \omega_0 \to \omega_0$

7. If $Fs\{x(t)\} = C_n$ then FS{x (-1/2 t)} is given by

(a) $C_{-n}\ with\ \omega_0 \to 2\omega_0$ (b) $C_{-n}\ with\ \omega_0 \to 1/2\omega_0$

(c) $1/C_n\ with\ \omega_0 \to 2\omega_0$ (d) $C_{-n}\ with\ \omega_0 \to \omega_0$

8. If $Fs\{x(t)\} = C_n$ then FS{x (t+3)} is given by

(a) $C_n e^{-3jnw_0} with\ \omega_0 \to \omega_0$ (b) $C_n e^{-3jnw_0}\ with\ \omega_0 \to 3\omega_0$

(c) $C_n e^{3jnw_0} with\ \omega_0 \to \omega_0$ (d) $C_n e^{3jnw_0}\ with\ \omega_0 \to 3\omega_0$

9. The time domain signal for following Fourier series coefficients

$$c_n = j\delta(n-1) - j\delta(n+1) + \delta(n-3) + \delta(n+3);\ \omega_0 = \pi \text{ is}$$

(a) 2cos3πt-sinπt (b) 2cos3πt-2sinπt (c) 2cosπt-2sinπt (d) cos3πt-sinπt

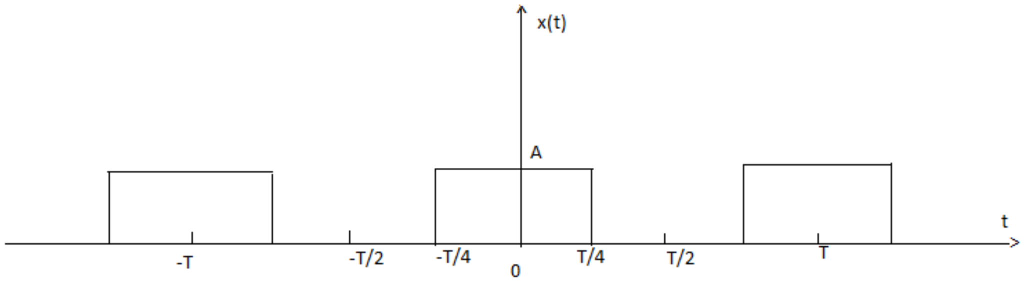
10. If £[$f(t)$] = F(s), then £[$f(t - T)$] =……

II Problems

1.For the continuous time periodic signal x (t) = 2+cos (2πt/3) +4sin (5πt/3), determine the fundamental frequency w_0 and the Fourier series coefficients c_n?

Ans: ω_0=π/3, C_{-5}= -2j ,C_{-2}= -0.5, C_0=2, C_5= 2j ,C_2= 0.5

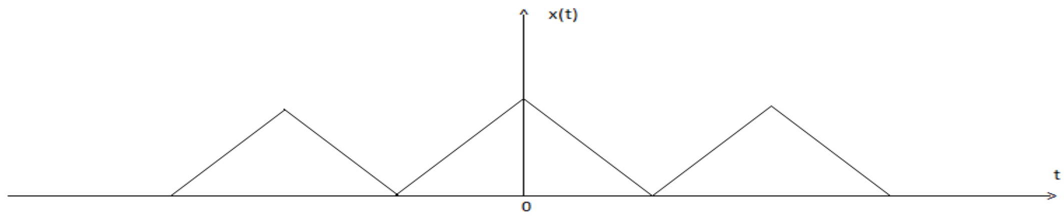
2. Obtain the Fourier components of the periodic rectangular wave form.



Ans:$\mathbf{x(t)=A/2+}\sum_{n=1}^{\infty}\frac{A}{2}\,sinc\left(\frac{n\omega T}{2}\right)cosn\omega t$

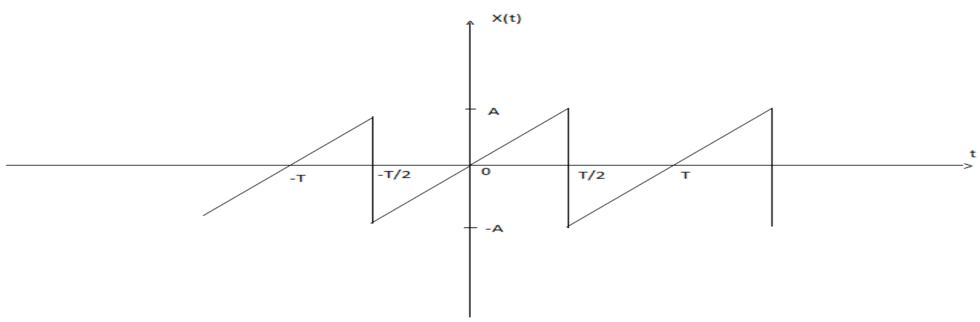
3. Find the Exponential Fourier series and plot frequency spectrum for the full wave rectified sine wave? Ans:$\mathbf{x(t)=}\sum_{n=-\infty}^{\infty}\frac{2A}{\pi(1-4n^2)}e^{j2nt}$

4. Find the trigonometric Fourier series for the waveform x(t) shown in the figure.



Ans: x(t)=$\mathbf{A/2+}\sum_{n=odd}^{\infty}\frac{4A}{\pi^2n^2}$ **cosnωt**

5. Obtain the trigonometric fourier series for the waveform.



Ans: $\mathbf{x(t)=}\ \sum_{n=1}^{\infty}\frac{2A}{n\pi}\ (-1)^{n+1}\,\sin n\frac{2\pi}{T}t$

6. Find the average power of the signal x(t) = $\cos^2(5000\pi t)\sin(20000\pi t)$. If this signal is transmitted through a telephone system which blocks dc and frequencies above 14 kHz, then compute the ratio of received power to transmitted power. Ans: **3/16 w, 5/6**

- By the use of Fourier Series we can represent any periodic function f(t) over the entire interval as a discrete sum of exponential functions.

Representation of an arbitrary function over the entire interval:

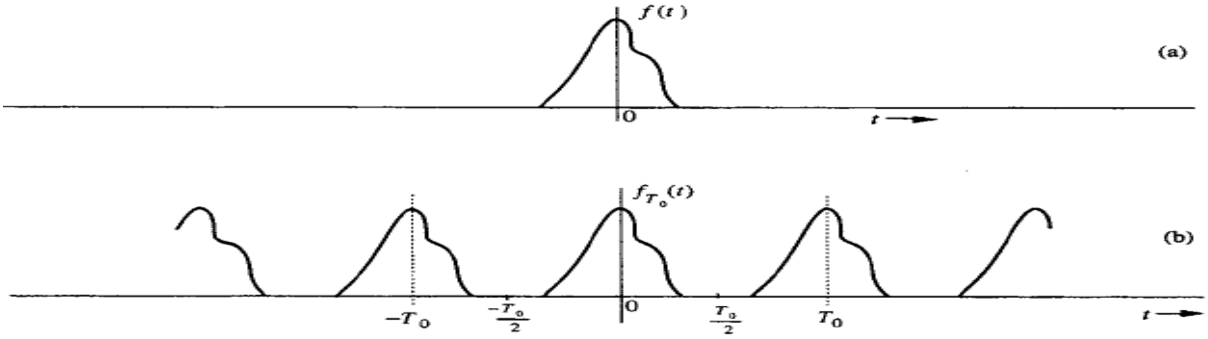


Fig. Construction of a periodic signal by periodic extension of f(t)

- This can be done by constructing a periodic function of period T. So that f(t) represents the first cycle of this periodic waveform.
- If T-> α, then the pulses in periodic function repeat after an infinite interval.

$$lt_{T->\infty} f_T(t) = f(t) \text{ ------------------------(1)}$$

- The exponential Fourier series of $f_T(t)$ is

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jntw_0}$$

Where $$w_0 = \frac{2\pi}{T}$$

$$F_n = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t) e^{jntw_0} dt \text{ ------------------------(2)}$$

- Let us consider $nw_0 = w_n$ then F_n is the function of w_n, and we shall denote F_n by $F_n(w_n)$

From Equation (2)

$$TF_n = \int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t) e^{jtw_n} dt$$

Let $TF_n(w_n) = F(w_n)$ --(3)

Then $f_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(w_n) w_0 e^{jtw_n}$ --(4)

- From the above equation $f_T(t)$ is expressed as a sum of exponential signals of frequencies

$w_1, w_2, \ldots\ldots\ldots\ldots w_n$. Let us represent the above equation (4) in graphical representation

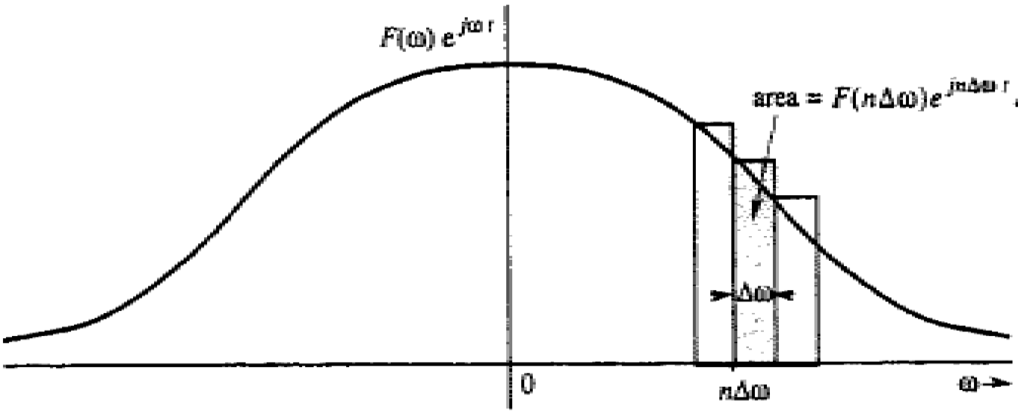


Fig. The Fourier series become Fourier integral as limit $T -> \infty$

- The discrete sum in equation (4) becomes the integral or the area under this curve. The curve now is continuous function of w and is given by $F(w)e^{jwt}$.
- Also $T -> \infty$**,** the function $f_T(t)$ -> f(t)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{jwt}dw \text{-----------------------------------}(5)$$

Where $$F(w) = \int_{-\infty}^{\infty} f(t)e^{-jwt}dt \text{--------------------------------------}(6)$$

<u>Existence of the Fourier Transform:</u>

The conditions for a function to have Fourier transform, called dirichilet's conditions

- f(t) is absolutely integrable over the interval $-\infty$ to ∞ that is $\int_{-\infty}^{\infty} f(t)dt < \infty$
- f(t) has a finite number of discontinuities in every finite interval.
- f(t) has a finite number of maxima and minima every finite interval.

Fourier transform of some useful functions:

	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\displaystyle\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Properties of continuous time Fourier transform:

1. Linearity property: If

$$f_1(t) \Longleftrightarrow F_1(\omega) \qquad \text{and} \qquad f_2(t) \Longleftrightarrow F_2(\omega)$$

then

$$a_1 f_1(t) + a_2 f_2(t) \Longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$$

2. Time shifting property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$f(t - t_0) \Longleftrightarrow F(\omega) e^{-j\omega t_0}$$

3. Frequency shifting property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$f(t) e^{j\omega_0 t} \Longleftrightarrow F(\omega - \omega_0)$$

4. Time Reversal property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$f(-t) \Longleftrightarrow F(-\omega)$$

5. Time Scaling property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$f(at) \Longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

6. Time Differentiation Property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$\frac{df}{dt} \Longleftrightarrow j\omega F(\omega)$$

7. Time Integration Property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$\int_{-\infty}^{t} f(\tau) d\tau \Longleftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

8. Time Convolution Property: If

$$f_1(t) \Longleftrightarrow F_1(\omega) \qquad \text{and} \qquad f_2(t) \Longleftrightarrow F_2(\omega)$$

then

$$f_1(t) * f_2(t) \Longleftrightarrow F_1(\omega)F_2(\omega)$$

9. Frequency Convolution Property: If

$$f_1(t) \Longleftrightarrow F_1(\omega) \qquad \text{and} \qquad f_2(t) \Longleftrightarrow F_2(\omega)$$

then

$$f_1(t)f_2(t) \Longleftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

10. Duality Property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$F(t) \Longleftrightarrow 2\pi f(-\omega)$$

11. Modulation Property: If

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$f(t)\cos\omega_0 t \Longleftrightarrow \frac{1}{2}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]$$

Parseval's Relation: If

$$f_1(t) \Longleftrightarrow F_1(\omega) \qquad \text{and} \qquad f_2(t) \Longleftrightarrow F_2(\omega)$$

then

$$\int_{-\infty}^{\infty} f_1(t)\, f_2^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(w)F_2^*(w)dw$$

Fourier Transform of a periodic function:

Let f(t) is a periodic function of period T. The Fourier transform of periodic function is

$$\Im[f(t)] = \sum_{-\infty}^{\infty} F_n \Im[e^{jntw_t}]$$

$$F(w) = \sum_{n=-\infty}^{\infty} F_n 2\pi\delta(w-nw_0)$$

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I) Objective Questions

1. What is Fourier Transform?

2. What are the limitations of Fourier Transform?

II) Descriptive Questions

1. Derive the Fourier Transform from exponential Fourier series.
2. State and prove properties of Fourier Transform.
3. Explain the Fourier transform of periodic signals

B. Question testing the ability of students in applying the concepts.

I) Multiple Choice Questions:

1. The Fourier Transform of a unit impulse function

(a) 1/w (b) 1 (c) w (d) 1/jw

2. The Fourier Transform of $e^{-at}u(t)$ is

(a) $\frac{1}{jw}$ (b) $\frac{1}{a+jw}$ (c) $\frac{1}{a-jw}$ (d) none

3. The Fourier Transform of DC signal x(t)=A is

(a) $2\pi A\delta(w)$ (b) $A\delta(w)$ (c) $2\pi A$ (d) Aw

4. The Fourier Transform of tx(t)

(a) $\frac{d}{dw}X(w)$ (b) $\frac{X(w)}{jw}$ (c) $j\frac{d}{dw}X(w)$ (d) None

5. The Fourier Transform of $x_1(t)x_2(t)$

(a) $X_1(w)*X_2(w)$ (b) $\frac{1}{2\pi}X_1(w)*X_2(w)$ (c) $2\pi X_1(w)*X_2(w)$ (d) $X_1(w)X_2(w)$

6. The Fourier Transform of unit step function __πδ(ω)+1/jω

7. The Fourier Transform of a signum function is

(a) $\frac{1}{jw}$ (b) $\frac{2}{jw}$ (c) $\frac{2}{w}$ (d) none

8. The Fourier Transform of $e^{-|t|}$

(a) $\frac{1}{1+w^2}$ (b) $\frac{2}{1+w^2}$ (c) $\frac{2}{1-w^2}$ (d) none

9. The Fourier Transform of $e^{-2t}u(t)$ is

(a) $\frac{1}{jw}$ (b) $\frac{1}{2+jw}$ (c) $\frac{1}{2-jw}$ (d) none

II) Problems:

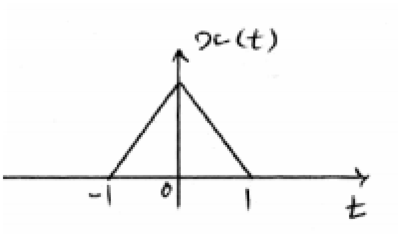
1. Find the Fourier Transform of the following signals

(i)$e^{-2t}\cos 5t\, u(t)$ (ii) $te^{-at}u(t)$ (iii) $\delta(t+2)+\delta(t+1)+\delta(t-1)+\delta(t-2)$ (iv) $e^{-a|t-2|}$ (v) $\frac{1}{t}$

Ans: **(i) $(2+j\omega)/[(2+j\omega)^2+5^2]$ (ii) $1/(a+j\omega)^2$ (iii) $2[\cos\omega+\cos 2\omega]$**

(iv) $[2a/(a^2+\omega^2)]\exp(-j2\omega)$ (v) $-j\pi\,\mathrm{sgn}(\omega)$

2. Find the Fourier Transform of the following signal



Ans: **$\mathrm{sinc}^2(\omega/2)$**

3. Determine the Fourier transform of a signum function Ans: **$2/j\omega$**

4. Find the Fourier Transform of

$$f(t)=1-t^2 \qquad 0 < t < 1$$
$$=0 \qquad \text{otherwise}$$

Ans: **$F(\omega) = 1/j\omega - 2/(j\omega)^3 + 2/(j\omega)^2\left(e^{-j\omega}+\frac{e^{-j\omega}}{j\omega}\right)$**

5. Find the Fourier Transform of the signal

$$f(t)=e^{-|t|} \qquad -2 < t < 2$$
$$=0 \qquad \text{otherwise}$$

Ans:**$F(\omega)=\frac{2}{1+\omega^2}-\frac{e^{-2}}{1+\omega^2}[2\cos 2\omega-2\omega\sin 2\omega]$**