Unit II – Indeterminate Beams

Contents:

- Concept of Analysis -Propped cantilever and fixed beams-fixed end moments and reactions

- Theorem of three moments – analysis of continuous beams – shear force and bending moment diagrams.

Unit II – Indeterminate Beams

- References:
- Punmia B.C.,"Theory of Structures" (SMTS) Vol II, Laxmi Publishing Pvt Ltd, New Delhi 2004.
- Rattan.S.S., "Strength of Materials", Tata McGraw Hill Education Pvt. Ltd., New Delhi, 2011.
- Rajput R.K., "Strength of Materials (Mechanics of Solids)", S.Chand & company Ltd., New Delhi, 2010.
- Ramamrutham S., “Theory of structures” Dhanpat Rai & Sons, New Delhi 1990.

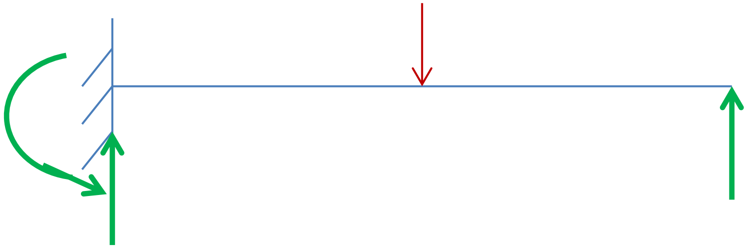
Indeterminate beams

- Statically determinate beams:
  - Cantilever beams
  - Simple supported beams
  - Overhanging beams

- Statically indeterminate beams:
  - Propped cantilever beams
  - Fixed beams
  - Continuous beams

Indeterminate beams

- Propped cantilever Beams:

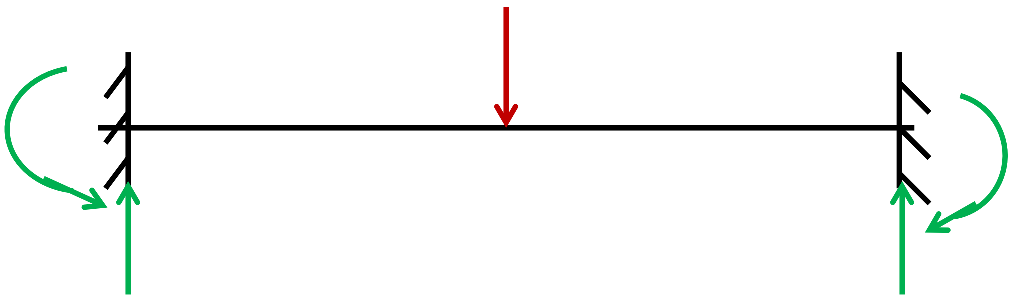


Degree of static indeterminacy=
 N0. of unknown reactions – static equations=3-2 = 1

Indeterminate beams

- Fixed beam:

A fixed beam is a beam whose end supports are such that the end slopes remain zero (or unaltered) and is also called a built-in or encaster beam.

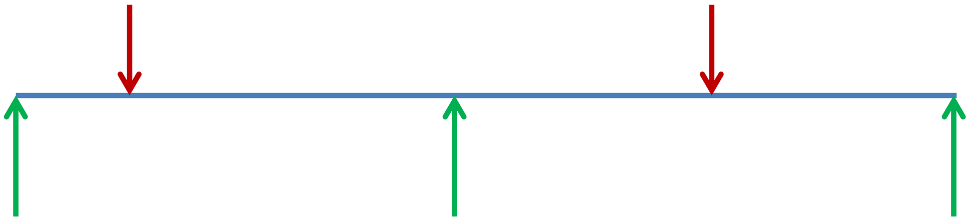


Degree of static indeterminacy=
N0. of unknown reactions – static equations=4-2 =2

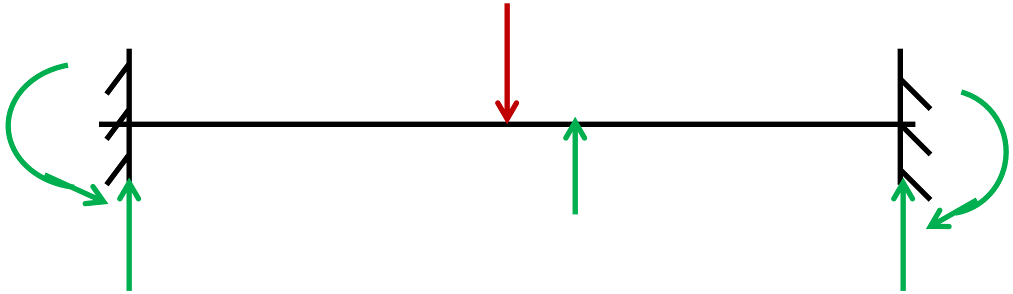
Indeterminate beams

Continuous beam: Continuous beams are very common in the structural design. For the analysis, theorem of three moments is useful.

A beam with more than 2 supports provided is known as continuous beam.



Degree of static indeterminacy=
N0. of unknown reactions – static equations=3-2 =1



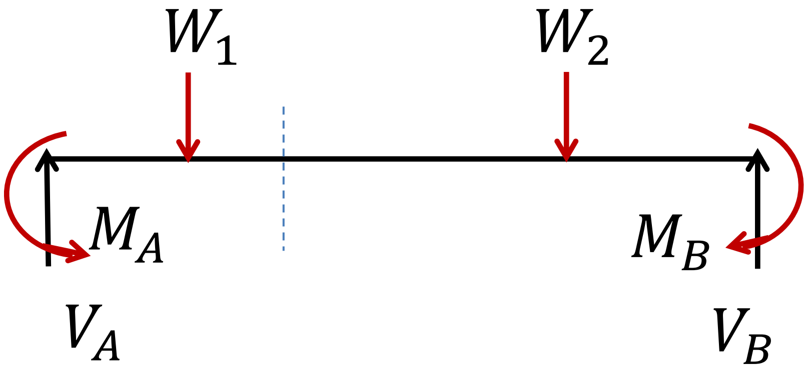
Degree of static indeterminacy=
N0. of unknown reactions – static equations=5-2 =3

Fixed Beams

- B.M. diagram for a fixed beam :

Figure shows a fixed beam AB carrying an external load system. Let V_A and V_B be the vertical reactions at the supports A and B.

Let M_A and M_B be the fixed end Moments.

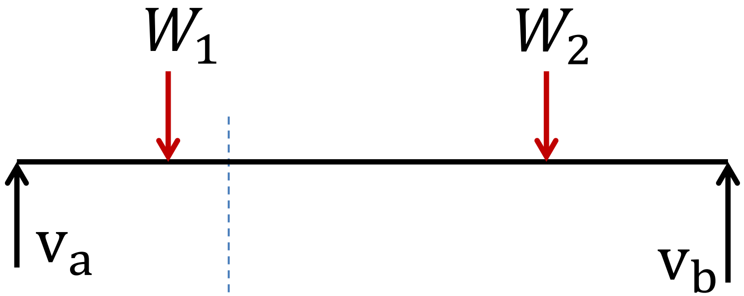


Fixed Beams

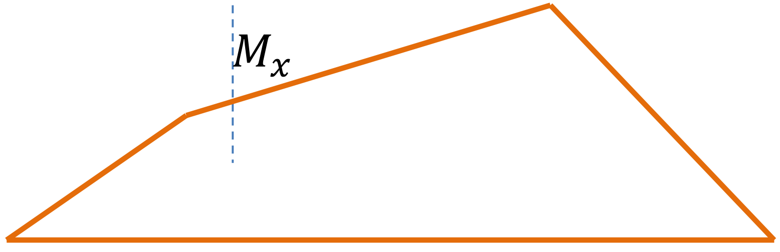
The beam may be analyzed in the following stages.

(i) Let us first consider the beam as Simply supported.

Let v_a and v_b be the vertical reactions at the supports A and B. Figure (ib) shows the bending moment diagram for this condition. At any section the bending moment M_x is a sagging moment.



(ia) Freely supported condition



(ib) Free B.M.D.

Fixed Beams

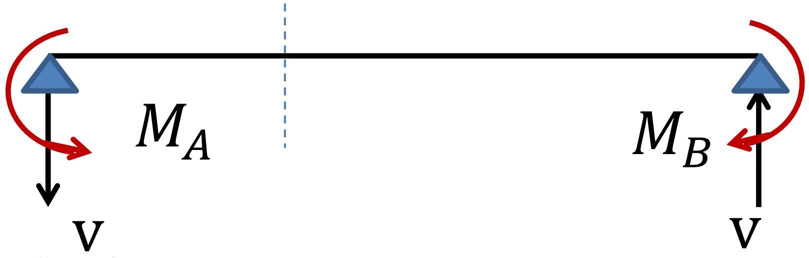
- (ii) Now let us consider the effect of end couples M_A and M_B alone.

Let v be the reaction at each end due to this condition.

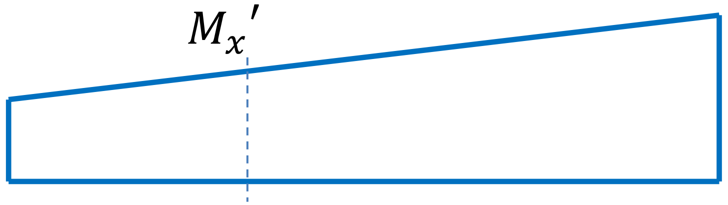
Suppose $M_B > M_A$.

Then $V = \frac{M_B - M_A}{L}$.

If $M_B > M_A$ the reaction V is upwards at B and downwards at A.



(iia) Effect of end couples



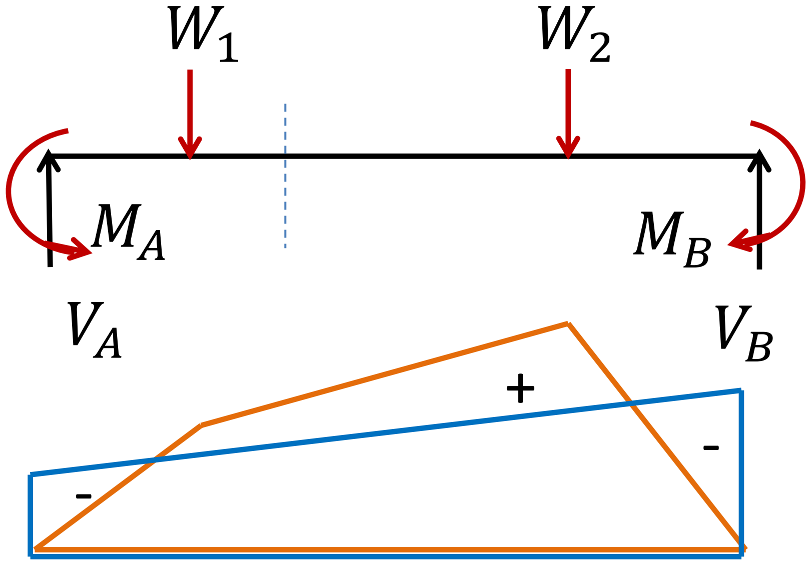
(iib) Fixed B.M.D.

Fig (iib). Shows the bending moment diagram for this condition.

At any section the bending moment $M_{x'}$ is hogging moment.

Fixed Beams

- Now the final bending moment diagram can be drawn by combining the above two B.M. diagrams as shown in Fig. (iiib)



(iiib) Resultant B.M.D.

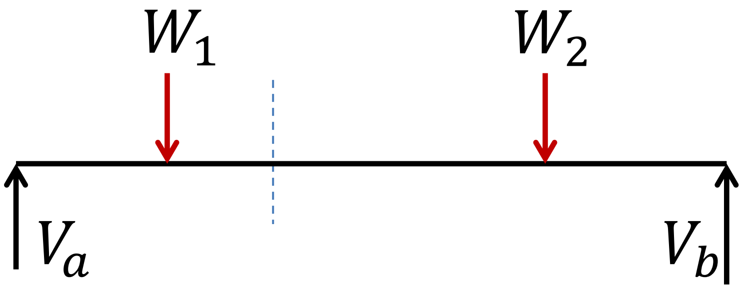
Now the final reaction $V_A = v_a - v$

and $V_B = v_b + v$

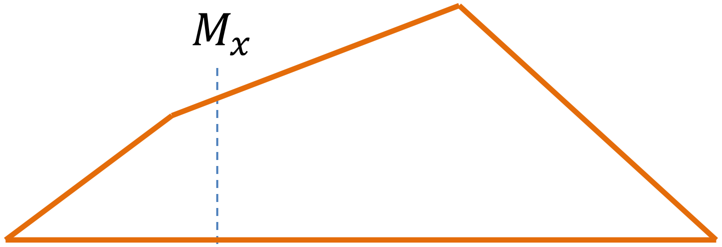
The actual bending moment at any section X, distance x from the end A is given by,

$$EI\,\frac{d^2y}{dx^2} = M_x - M_x{}'$$

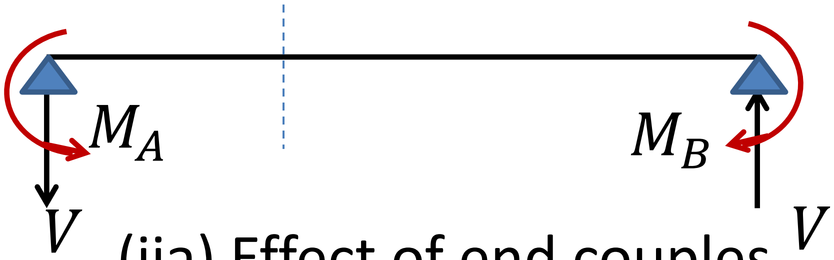
Fixed Beams



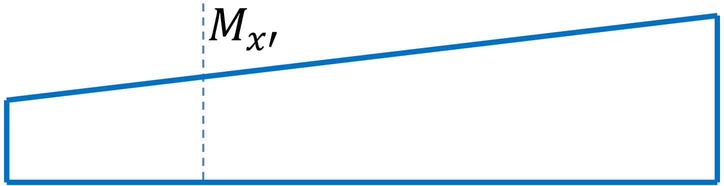
(ia) Freely supported condition



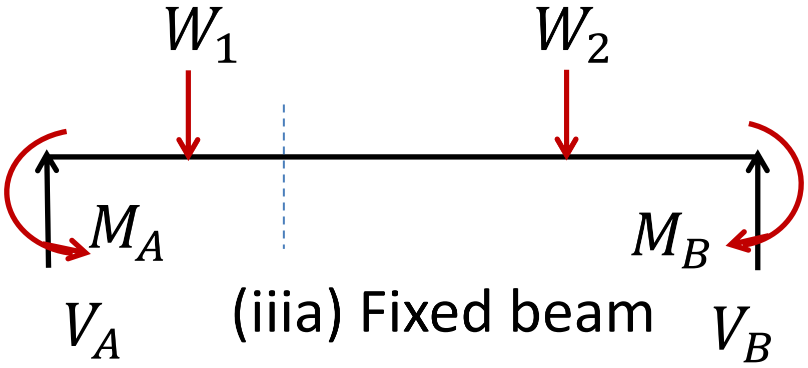
(ib) Free B.M.D.



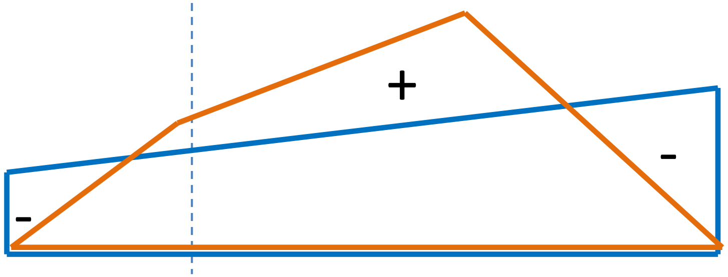
(iia) Effect of end couples



(iib) Fixed B.M.D.



(iiia) Fixed beam



(iiib) Resultant B.M.D.

Fixed Beams

$$EI\ \frac{d^2y}{dx^2} = M_x - M_x{}'$$

- Integrating, we get,
- $EI\left[\frac{dy}{dx}\right]_0^l = \int_0^l M_x dx - \int_0^l M_x' dx$
- But at x=0, $\frac{dy}{dx} = 0$

  and at $x = l, \frac{dy}{dx} = 0$

Further $\int_0^l M_x dx$ = area of the Free BMD = a

$$\int_0^l M_x' dx = \text{area of the fixed B.M.D} = a'$$

Substituting in the above equation, we get,

$0 = a - a'$

$$\therefore a = a'$$

Fixed Beams

$$a = a'$$

∴ Area of the free B.M.D. = Area of the fixed B.M.D.

Again consider the relation,

$$EI\,\frac{d^2y}{dx^2} = M_x - M_x{}'$$

$$Multying\ by\ x\ we\ get,$$

$$EIx\,\frac{d^2y}{dx^2} = M_x x - M'_x x$$

- Integrating we get,
- $\int_0^l EIx\,\frac{d^2y}{dx^2} = \int_0^l M_x x\,dx\ - \int_0^l M'_x x\,dx$
- $\therefore\ EI\left[x\frac{dy}{dx} - y\right]\begin{matrix} l \\ 0 \end{matrix} =$ a$\bar{x}$-a'$\bar{x}'$
- Where $\bar{x}$= distance of the centroid of the free B.M.D. from A.

  and $\bar{x}'$= distance of the centroid of the fixed B.M.D. from A.

Fixed Beams

- Further at x=0, y=0 and $\frac{dy}{dx} = 0$
- and at x=l, y=0 and $\frac{dy}{dx} = 0.$
- Substituting in the above relation, we have

$$0 = a\bar{x}\text{-}a'\bar{x}'$$

$$a\bar{x}\text{=}a'\bar{x}'$$

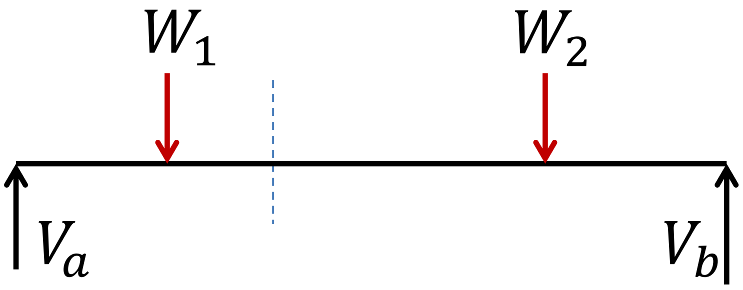
or $\bar{x} = \bar{x}$

∴ The distance of the centroid of the free B.M.D . From A= The distance of the centroid of the fixed B.M.D. from A.

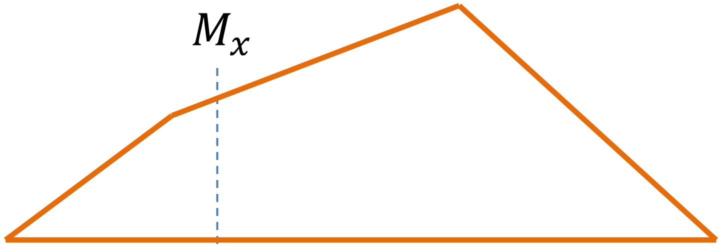
$$\therefore a = a'$$

$$\bar{x} = \bar{x}$$

Fixed Beams



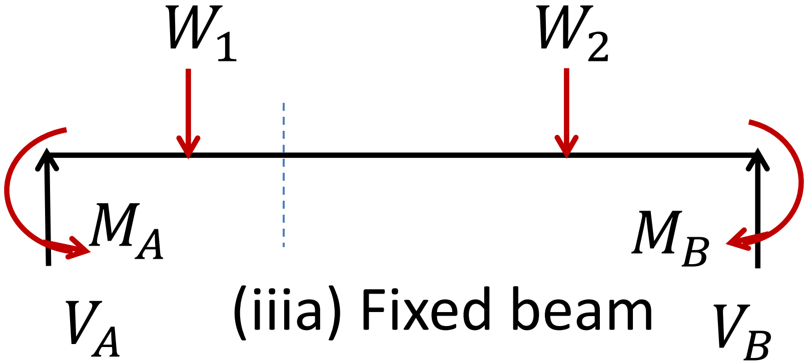
(ia) Freely supported condition



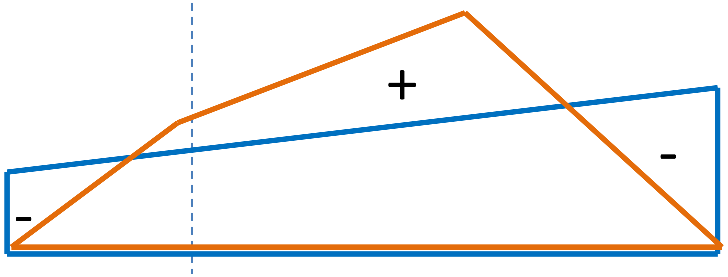
(ib) Free B.M.D.

(iia) Effect of end couples

(iib) Fixed B.M.D.



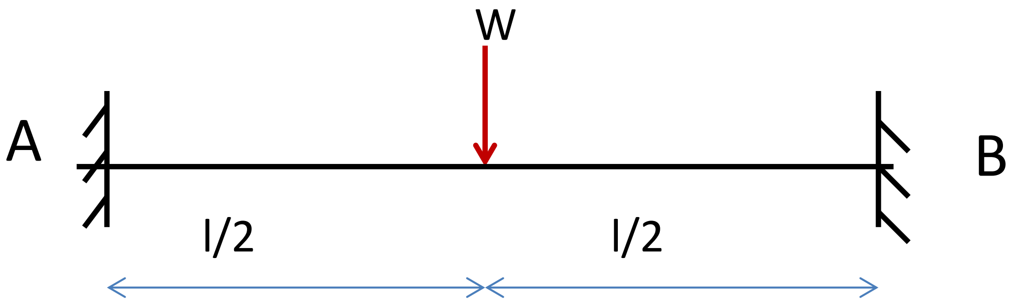
(iiia) Fixed beam



(iiib) Resultant B.M.D.

Fixed beam problems

- Find the fixed end moments of a fixed beam subjected to a point load at the center.

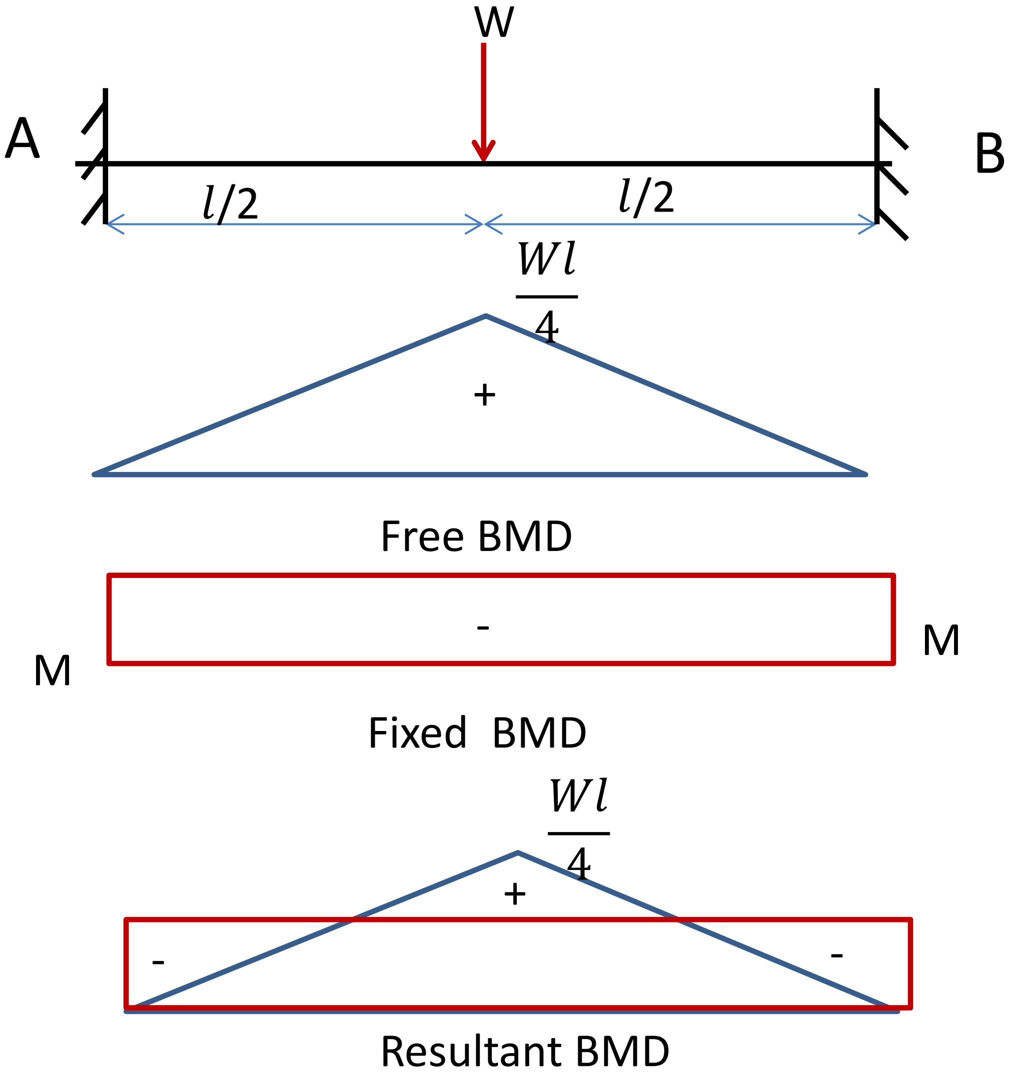


Fixed beam problems

- $A' = A$

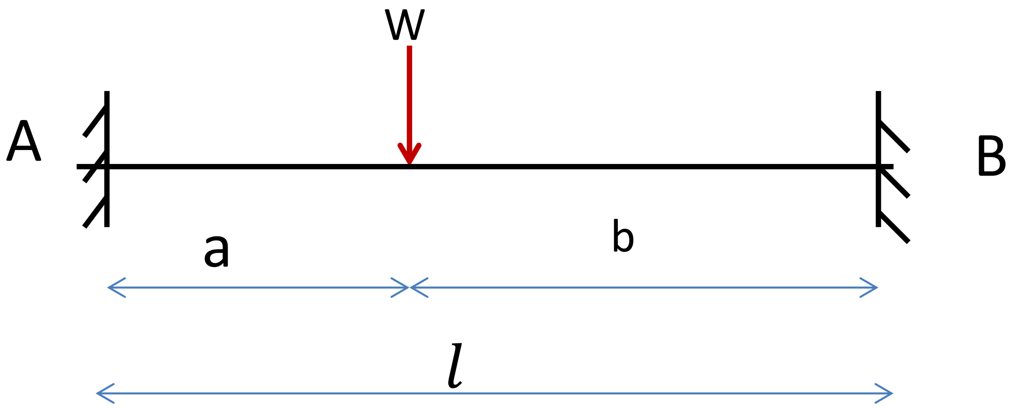
$$M \times l = \frac{1}{2} \times l \times \frac{Wl}{4}$$

$$M = \frac{Wl}{8} = M_A = M_B$$



Fixed beam problems

- Find the fixed end moments of a fixed beam subjected to a eccentric point load.



Fixed beam problems

- $A' = A$

$$\frac{M_A + M_B}{2} \times l = \frac{1}{2} \times l \times \frac{Wab}{l}$$

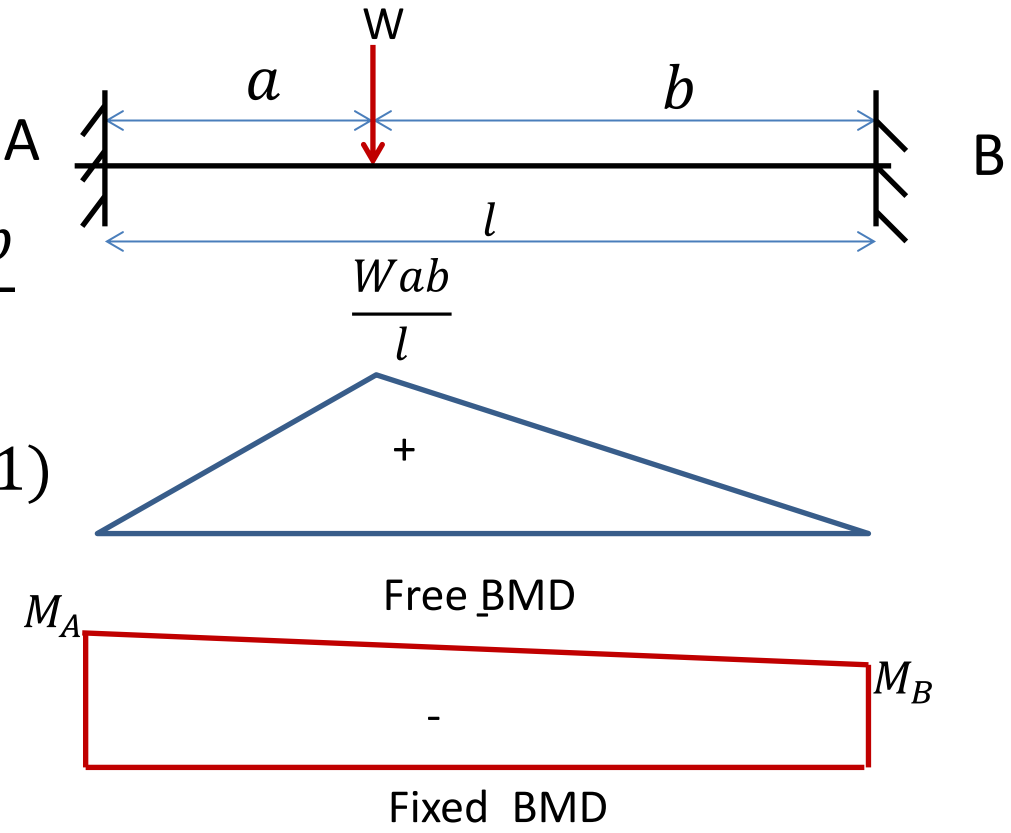
$$M_A + M_B = \frac{Wab}{l} ----(1)$$

- $x' = x$

$$\frac{M_A + 2M_B}{M_A + M_B} \times \frac{l}{3} = \frac{l + a}{3}$$

$$M_B = M_A \times \frac{a}{l - a}$$

$$M_B = \frac{a}{b} ---(2)$$



Fixed beam problems

$$M_A + M_B = \frac{Wab}{l} ----(1)$$

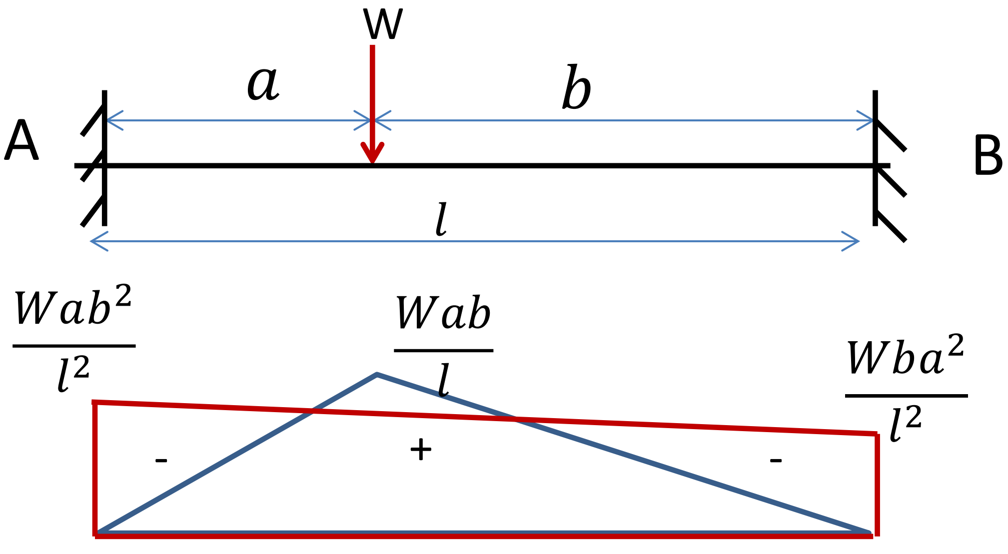
$$M_B = M_A \times \frac{a}{b} ---(2)$$

By substituting (2) in (1),

$$M_A = \frac{Wab^2}{l^2}$$

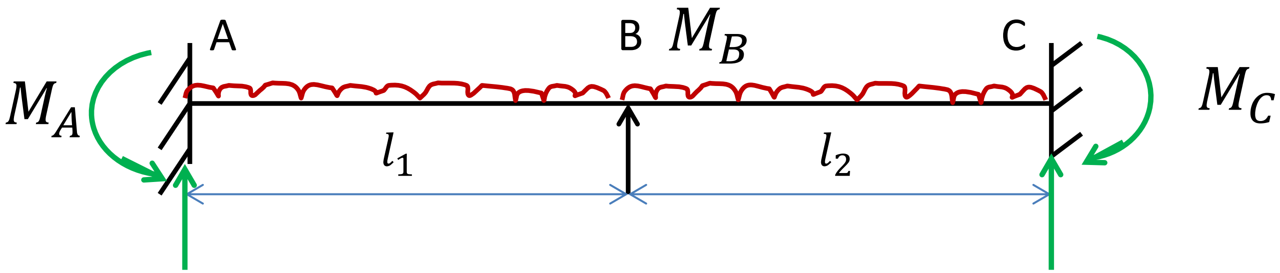
From (2),

$$M_B = \frac{Wba^2}{l^2}$$



Resultant BMD

Clapeyron's theorem of three moments



- As shown in above Figure, AB and BC are any two successive spans of a continuous beam subjected to an external loading.

- If the extreme ends A and C fixed supports, the support moments M_A, M_B and M_C at the supports A, B and C are given by the relation,

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

Clapeyron’s theorem of three moments (contd…)

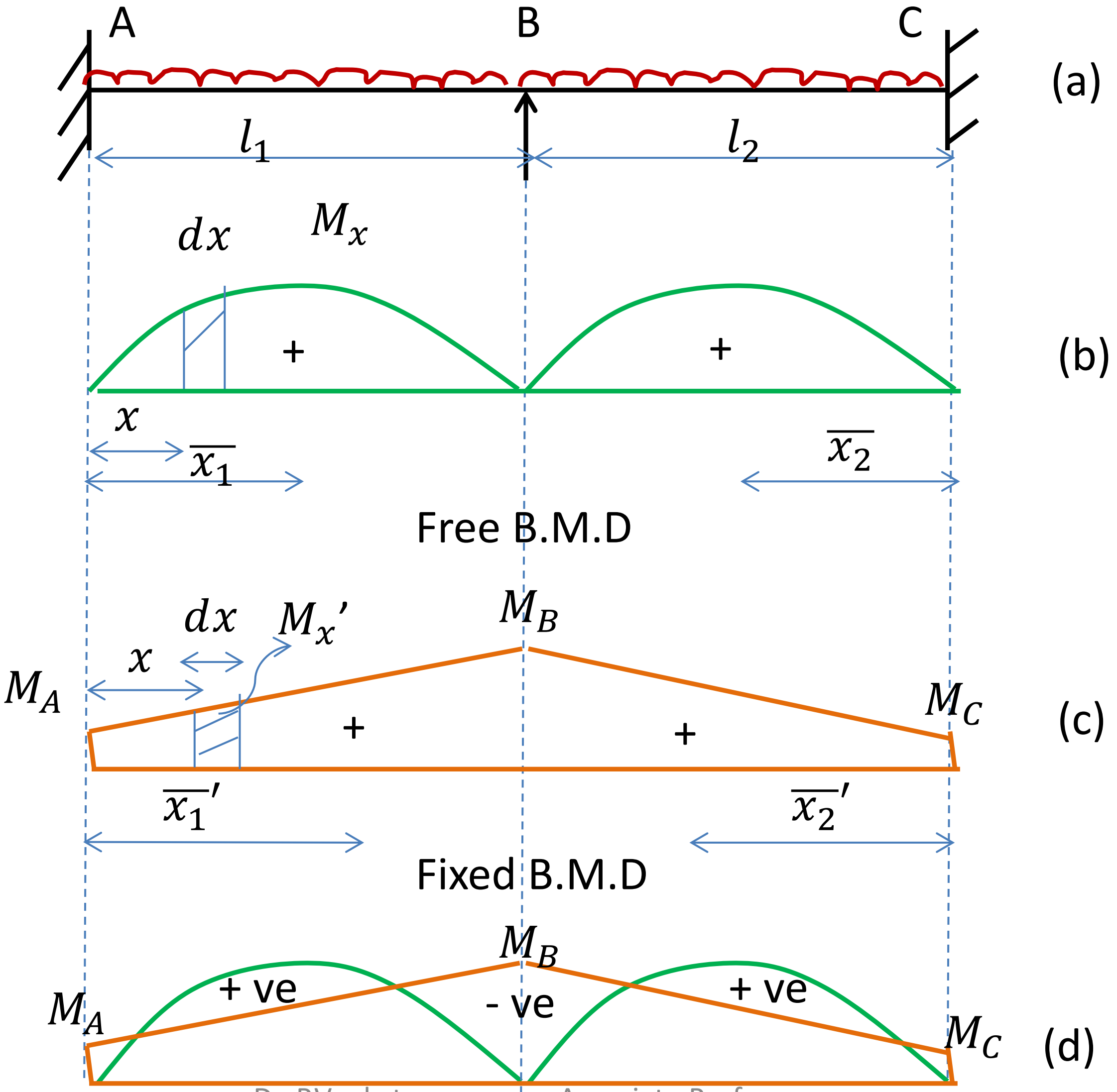
$$M_A l_1 + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

- Where,
- a_1 = area of the free B.M. diagram for the span AB.
- a_2 = area of the free B.M. diagram for the span BC.

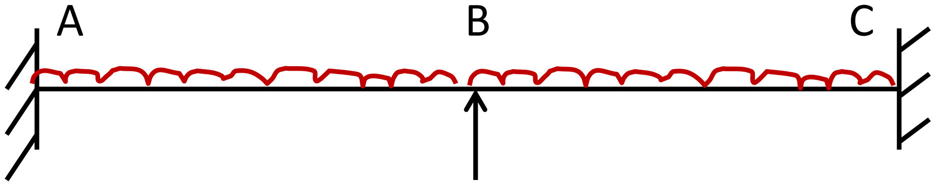
- $\overline{x_1}$= Centroidal distance of free B.M.D on AB from A.

- $\overline{x_2}$= Centroidal distance of free B.M.D on BC from C.

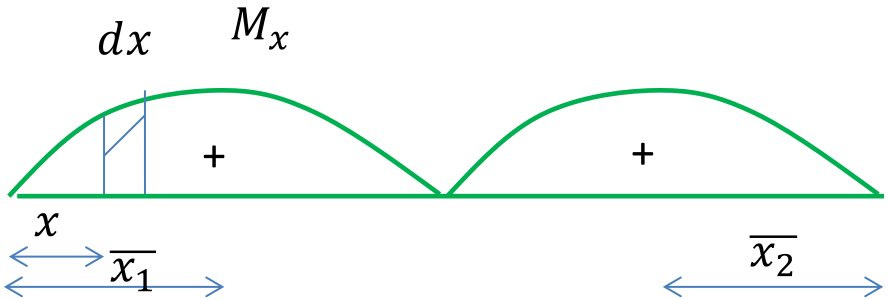
Clapeyron’s theorem of three moments (contd…)



Clapeyron’s theorem of three moments (contd…)

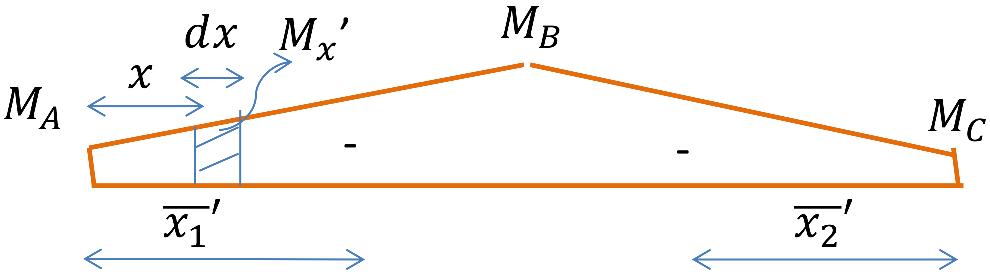


(a)The given beam



Free B.M.D

(b) Free B.M.D.



Fixed B.M.D

(c) Fixed B.M.D.

Clapeyron’s theorem of three moments (contd…)

- Consider the span AB:
- Let at any section in AB distant x from A the free and fixed bending moments be M_x and $M_x{}'$ respectively.

- Hence the net bending moment at the section is given by

$$EI\frac{d^2y}{dx^2} = M_x - M_x{}'$$

- Multiplying by x, we get

$$EIx\frac{d^2y}{dx^2} = M_x x - M_x{}'x$$

Clapeyron's theorem of three moments (contd…)

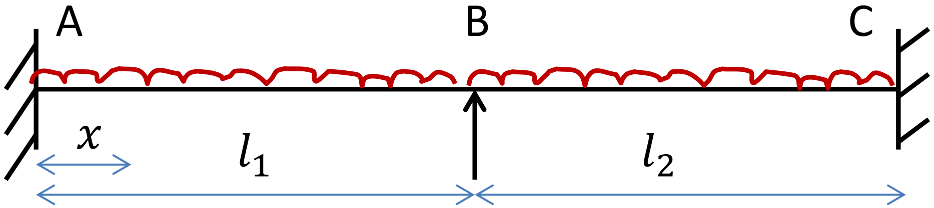
- $EIx\frac{d^2y}{dx^2} = M_x x - M_x{}' x$

- Integrating from $x = 0\ to\ x = l_1$, we get,

$$EI\int_0^{l_1} x\frac{d^2y}{dx^2} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x{}' x\,dx$$

$$EI\left[x.\frac{dy}{dx} - y\right]_0^{l_1} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x{}' x\,dx \qquad ---(1)$$

Clapeyron's theorem of three moments (contd…)



- But it may be such that

  At $x = 0$, deflection $y = 0$

- $At\ x = l_1, y = 0;\ and\ slope\ at\ B\ for AB, \frac{dy}{dx} = \theta_{BA}$

- $\int_0^{l_1} M_x x\, dx = a_1 \overline{x_1}$ = Moment of the free B. M. D. on AB about A .

- $\int_0^{l_1} M_x{}' x\, dx = a_1' \ \overline{x_1}{}' =$ Moment of the fixed B. M. D. on AB about A.

Clapeyron's theorem of three moments (contd…)

$$EI\left[x.\frac{dy}{dx}-y\right]_0^{l_1}=\int_0^{l_1} M_x x\,dx-\int_0^{l_1} M_x{}'x\,dx\quad --(1)$$

- Therefore the equation (1) is simplified as,

$$EI\ [l_1\theta_{BA}-0]\ = a_1\overline{x_1}\ \ -a_1'\overline{x_1}'.$$

But a_1' = area of the fixed B.M.D. on AB = $\frac{(M_A+M_B)}{2}l_1$

$\overline{x_1}'$ = $\text{Centroid of the fixed B. M. D. from A}$ = $\frac{(M_A+2M_B)}{M_A+M_B}\frac{l_1}{3}$

Clapeyron’s theorem of three moments (contd…)

- Therefore,

$$a_1^{'}\overline{x_1}^{'} = \frac{(M_A + M_B)}{2} l_1 \times \left(\frac{M_A + 2M_B}{M_A + M_B}\right)\frac{l_1}{3} = (M_A + 2M_B)\frac{l_1^2}{6}$$

$$EI\ l_1\theta_{BA} = a_1\overline{x_1} - (M_A + 2M_B)\frac{l_1^2}{6}$$

$$6EI\ \theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - (M_A + 2M_B)l_1 \qquad ----(2)$$

Similarly by considering the span BC and taking C as origin it can be shown that,

$$6EI\ \theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - (M_C + 2M_B)l_2 \qquad ----(3)$$

$\theta_{BC} =$ slope for span CB at B

Clapeyron’s theorem of three moments (contd…)

- But $\theta_{BA} = -\theta_{BC}$ as the direction of x from A for the span AB, and from C for the span CB are in opposite direction.
- And hence, $\theta_{BA} + \theta_{BC} = 0$

$$\boldsymbol{6EI\,\theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - (M_A + 2M_B)l_1 \qquad ----(2)}$$

$$\boldsymbol{6EI\,\theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - (M_C + 2M_B)l_2 \qquad ----(3)}$$

- Adding equations (2) and (3), we get

$$\boldsymbol{EI\,\theta_{BA} + 6EI\,\theta_{BC} = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - (M_A + 2M_B)l_1 - (M_C + 2M_B)l_2}$$

$$\boldsymbol{6EI(\theta_{BA} + \theta_{BC}) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - [M_Al_1 + 2M_B(l_1 + l_2) + M_Cl_2]}$$

Clapeyron’s theorem of three moments (contd…)

$$\mathbf{0 = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - [M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2]}$$

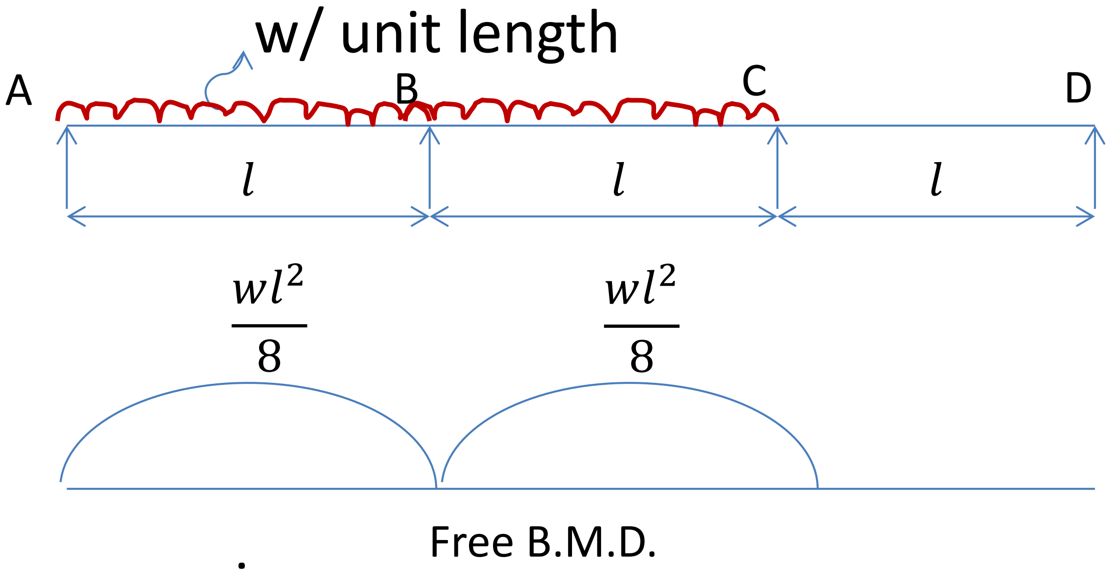
$$\mathbf{[M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2] = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}}$$

Problems

- A continuous beam of three equal span is simply supported over two supports. It is loaded with a uniformly distributed load of w/unit length, over the two adjacent spans only. Using the theorem of three moments, find the support moments and sketch the bending moment diagram. Assume EI constant.

Problems

- Solution:



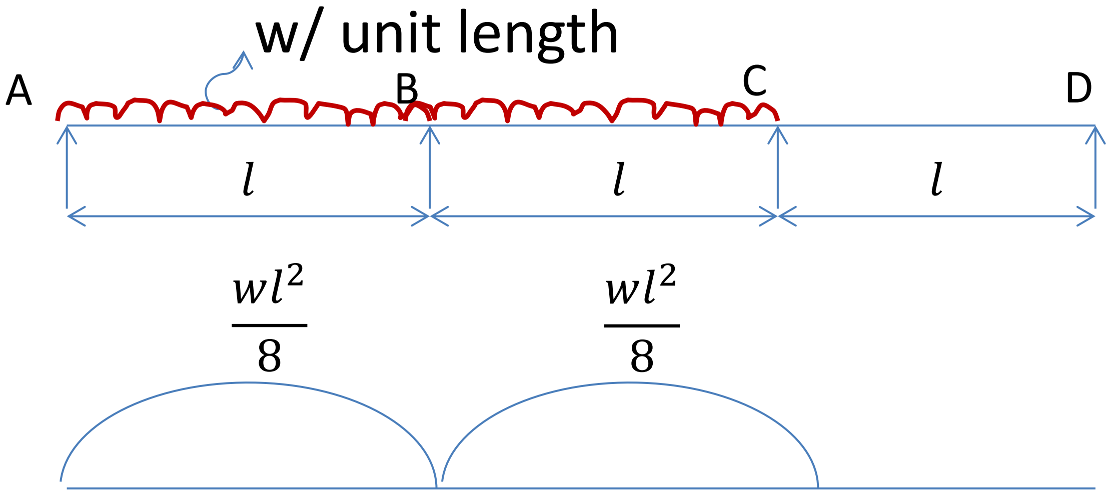
- The theorem of three moments equation for two spans is,

$$M_A(l_1) + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

Apply the theorem of three moment equation for spans AB and BC is,

$$M_A(l) + 2M_B(l + l) + M_C(l) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

Problems



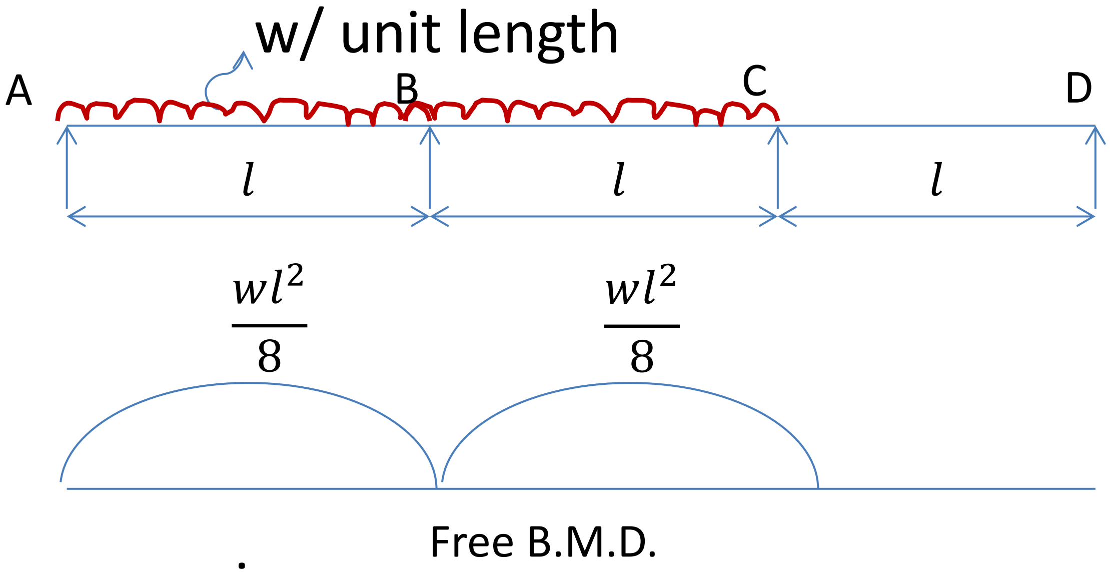
Free B.M.D.

- Solution:
- $a_1 = \frac{2}{3} \times l \times \frac{wl^2}{8}$

  $= \frac{1}{12} wl^3$
- $\overline{x_1} = \frac{l}{2}$
- $a_2 = \frac{2}{3} \times l \times \frac{wl^2}{8}$
- $\overline{x_2} = \frac{l}{2}$
- $\overset{0}{\cancel{M_A}}(l) + 2M_B(l + l) + M_C(l) = \frac{6 \times \frac{1}{12} wl^3 \times \frac{l}{2}}{l} + \frac{6 \times \frac{1}{12} wl^3 \times \frac{l}{2}}{l}$
- $4M_B + M_C = \frac{wl^2}{2} -----(1)$

Problems

- Solution:



- The theorem of three

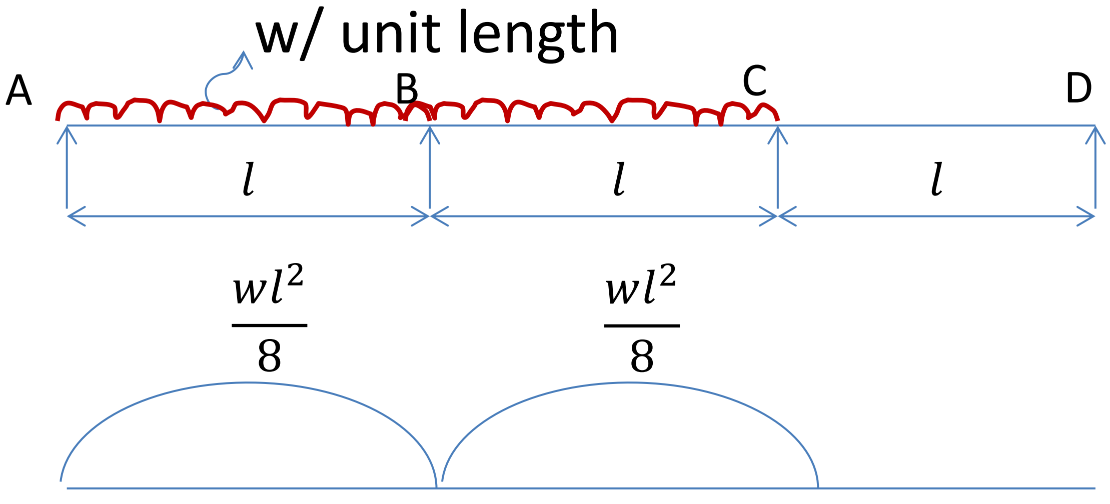
moments equation for two spans is,

$$M_A(l_1) + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

Apply the theorem of three moment equation for spans BC and CD is,

$$M_B(l) + 2M_C(l + l) + M_D(l) = \frac{6a_1\overline{x_1}}{l_1} + 0$$

Problems



Free B.M.D.

- Solution:
- $a_1 = \frac{2}{3} \times l \times \frac{wl^2}{8}$

  $=\frac{1}{12} wl^3$
- $\overline{x_1} = \frac{l}{2}$
- $M_B(l) + 2M_C(l+l) + \cancelto{0}{M_D}(l) = \frac{6 \times \frac{1}{12} wl^3 \times \frac{l}{2}}{l} + 0$
- $M_B + 4M_C = \frac{wl^2}{4} - - - - - -(2)$

Problems

- $4M_B + 16M_C = \mathrm{w}l^2 ------(2) \times 4$

$$4M_B + M_C = \frac{wl^2}{2} ------(1)$$

$$15M_C = \frac{wl^2}{2} ------(2) \times 4 - (1)$$

$$M_C = \frac{wl^2}{30}$$

Substitute $M_C = \frac{wl^2}{30}$ in equation (2),

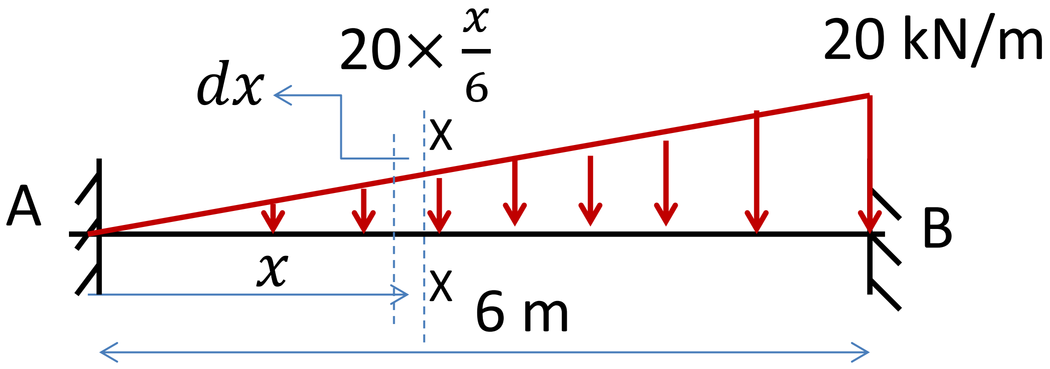
$$M_B + 4 \times \frac{wl^2}{30} = \frac{wl^2}{4}$$

$$M_B = \frac{7wl^2}{60}$$

Fixed beam Problems

- A fixed beam AB of span 6 m carries uniformly varying load of intensity zero at A and 20 kN/m at B. Find the fixed end moments and draw the B.M. and S.F. diagrams for the beam.

Problems



Consider any section XX distant x from the end A, the intensity of loading at XX = $\frac{wx}{L} = \frac{20x}{6}$

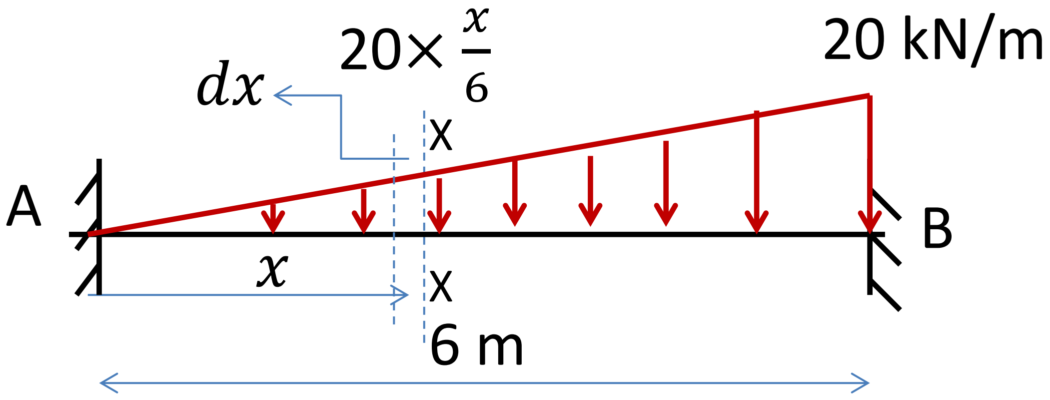
Hence the load acting for an elemental distance $dx = \frac{20x}{6}dx$

Due to this elemental load the fixed moments are as follows:

$$dM_a = \frac{\mathrm{W}ab^2}{L^2} \text{(Formula is derived from first principles)}$$

$$= \frac{\frac{20x}{6}dx \times x \times (6-x)^2}{6^2} = \frac{20x^2(6-x)^2 dx}{6^3}$$

Problems



and

$$dM_b = \frac{Wba^2}{L^2}\text{(Formula is derived from basic principles)}$$

$$= \frac{\frac{20x}{6}dx\times(6-x)\times(x)^2}{6^2} = \frac{20x^3(6-x)dx}{6^3}$$

Taking fixing moment at A,

$$M_A = \int_0^l dM_a = \int_0^6 \frac{20}{216}x^2(6-x)^2dx$$

Problems

$$M_A = \frac{20}{216}\int_0^6 x^2(36 + x^2 - 12x)dx$$

$$= \frac{20}{216}\left[\frac{36x^3}{3} + \frac{x^5}{5} - \frac{12x^4}{4}\right]\Bigg|_0^6$$

$$= \frac{20}{216}\left[\frac{36 \times 6^3}{3} + \frac{6^5}{5} - \frac{12 \times 6^4}{4}\right]$$

$$\therefore M_A = 24 \text{ kNm}$$

Problems

$$M_B = \int_0^l dM_B = \int_0^6 \frac{20}{6^3} x^3 (6 - x) dx$$

$$= \frac{20}{216}\left[\frac{x^4}{4} \times 6 - \frac{x^5}{5}\right] \Big|_0^6$$

$$= \frac{20}{216}\left[\frac{6^4 \times 6}{4} - \frac{6^5}{5}\right]$$

$$\therefore M_B = 36 \text{ kNm}$$

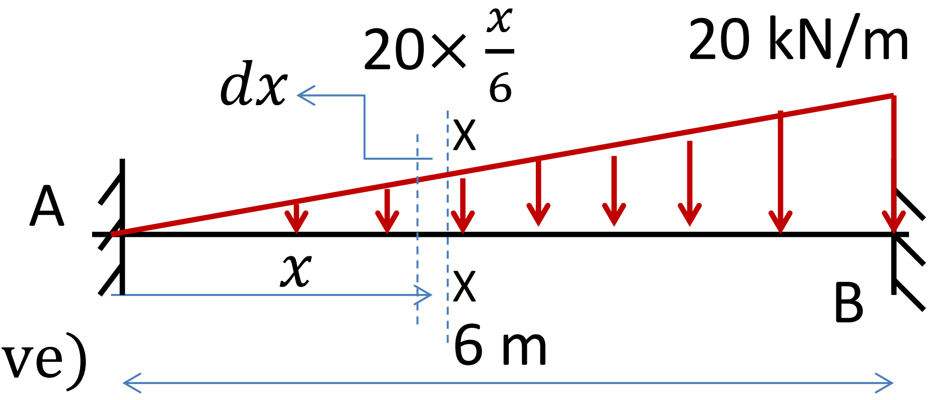
Problems

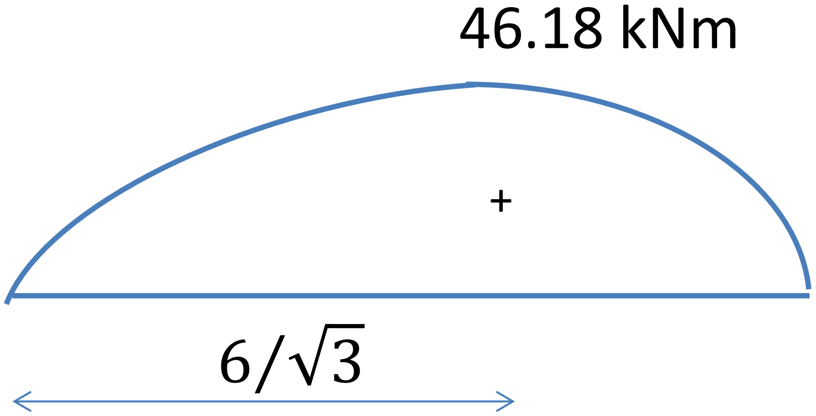
Free BMD:

$$M_{max} = \frac{wl^2}{9\sqrt{3}} = \frac{20 \times 6^2}{9\sqrt{3}}$$

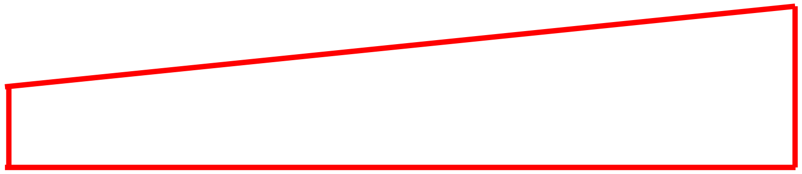
$$= 46.18 \text{ kNm (Cubic parabolic curve)}$$

Will occur at $6/\sqrt{3}$ m from left end A.





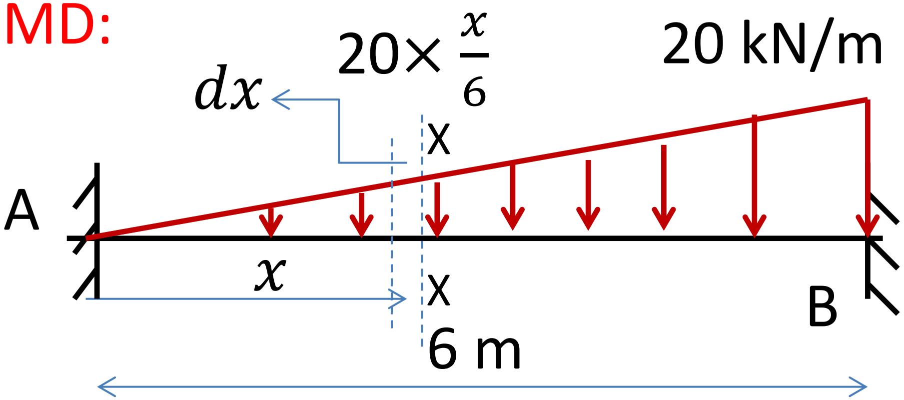
Free BMD

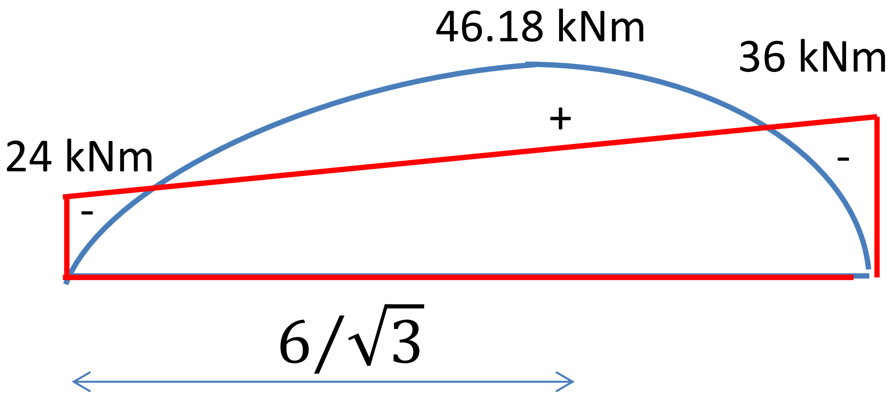


Fixed BMD

Problems

Resultant BMD:

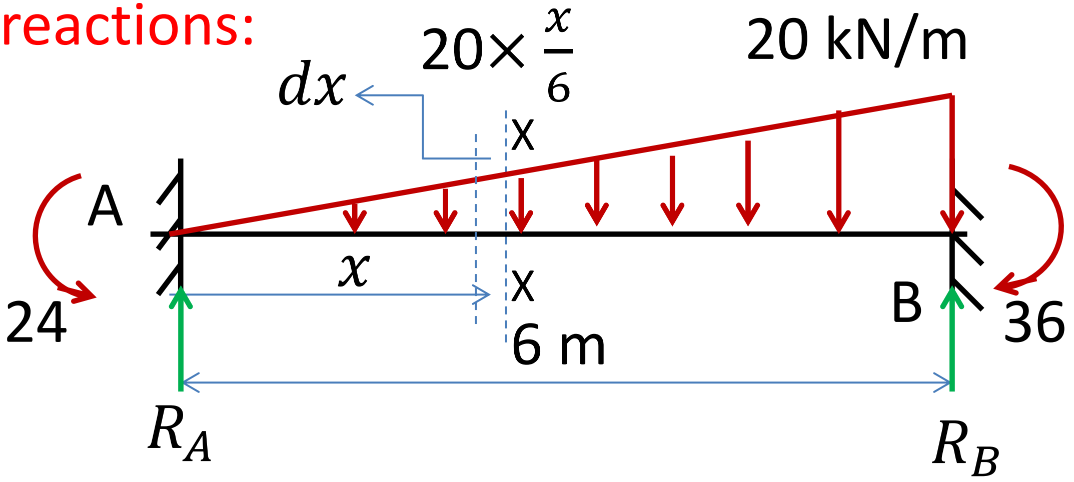




Resultant BMD

Problems

Calculation of support reactions:



$$\sum M_A = 0$$

$$R_B \times 6 + 24 = 36 + \frac{1}{2} \times 6 \times 20 \times \frac{2}{3} \times 6$$

$$R_B = \frac{252}{6} = 42 \text{ kN}$$

$$R_A + 42 = \frac{1}{2} \times 6 \times 20$$

$$R_A = 60 - 42 = 18 \text{ kN}$$

Problems

SFD:

S.F. @ A = +18 kN

S.F. @ B = -42 kN

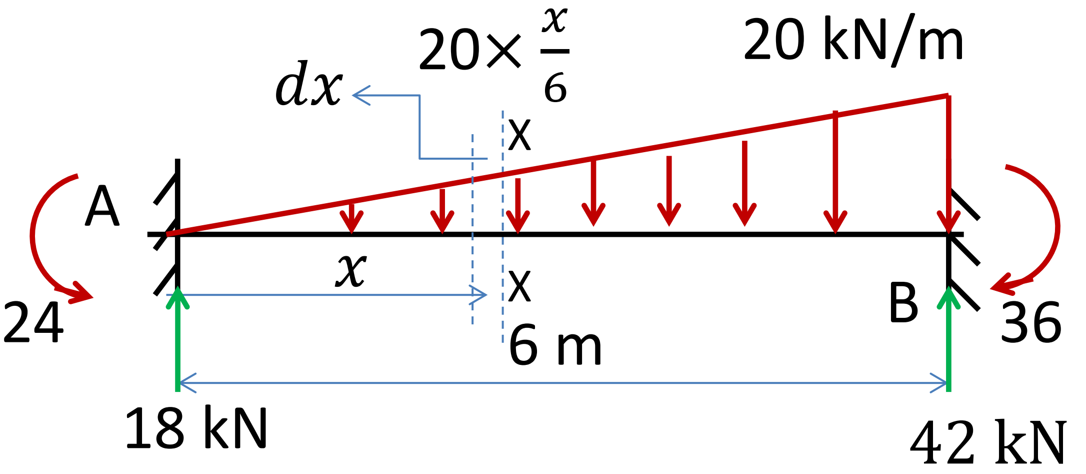
SFD between A and B is a parabola.

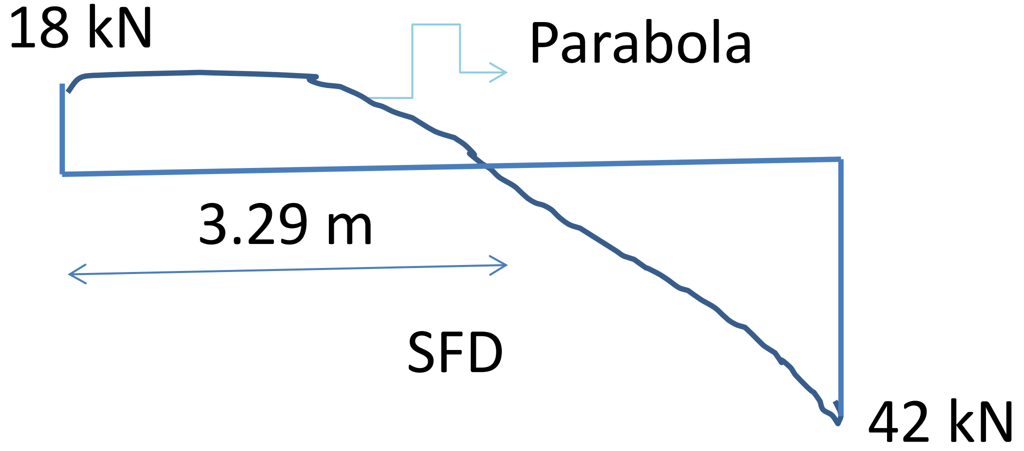
S.F. @ XX = 0

$$18 - \frac{1}{2} \times x \times 20 \times \frac{x}{6} = 0$$

$$18 = \frac{10x^2}{6}$$

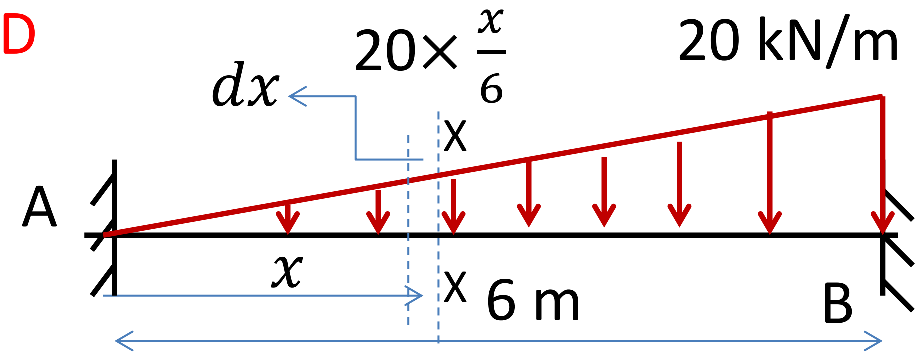
$$x = 3.29\,m$$

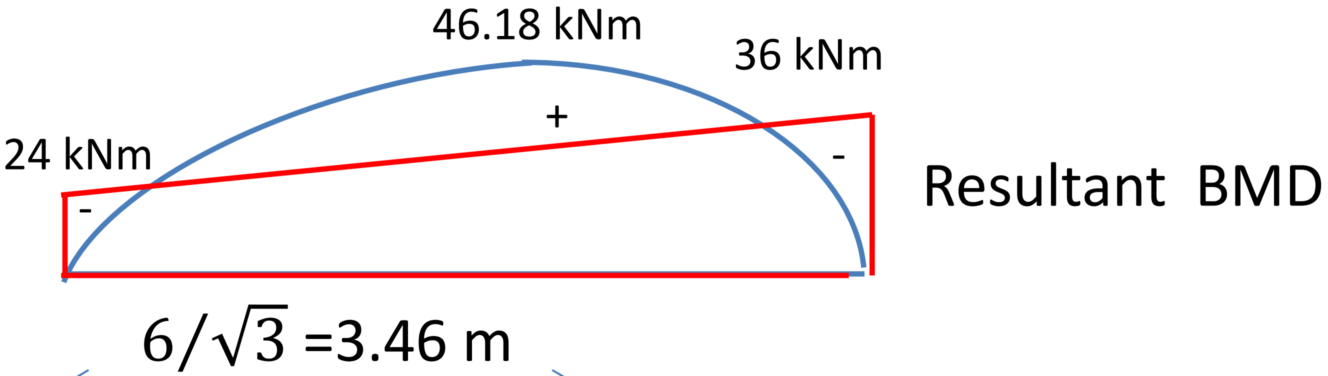


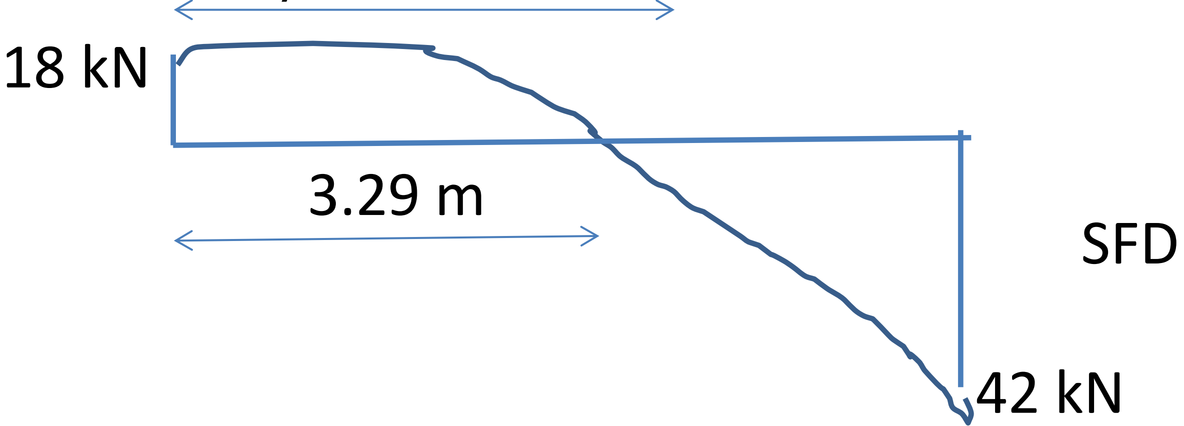


Problems

Resultant BMD & SFD:







Fixed Beams - Problems

- A beam AB of 12m span has fixed ends. It carries a downward load of 120 kN at 4 m from end A and an upward load of 80 kN at 6 m from end B. Calculate the fixed end moments and draw the bending moment diagram.

Fixed Beams - Problems

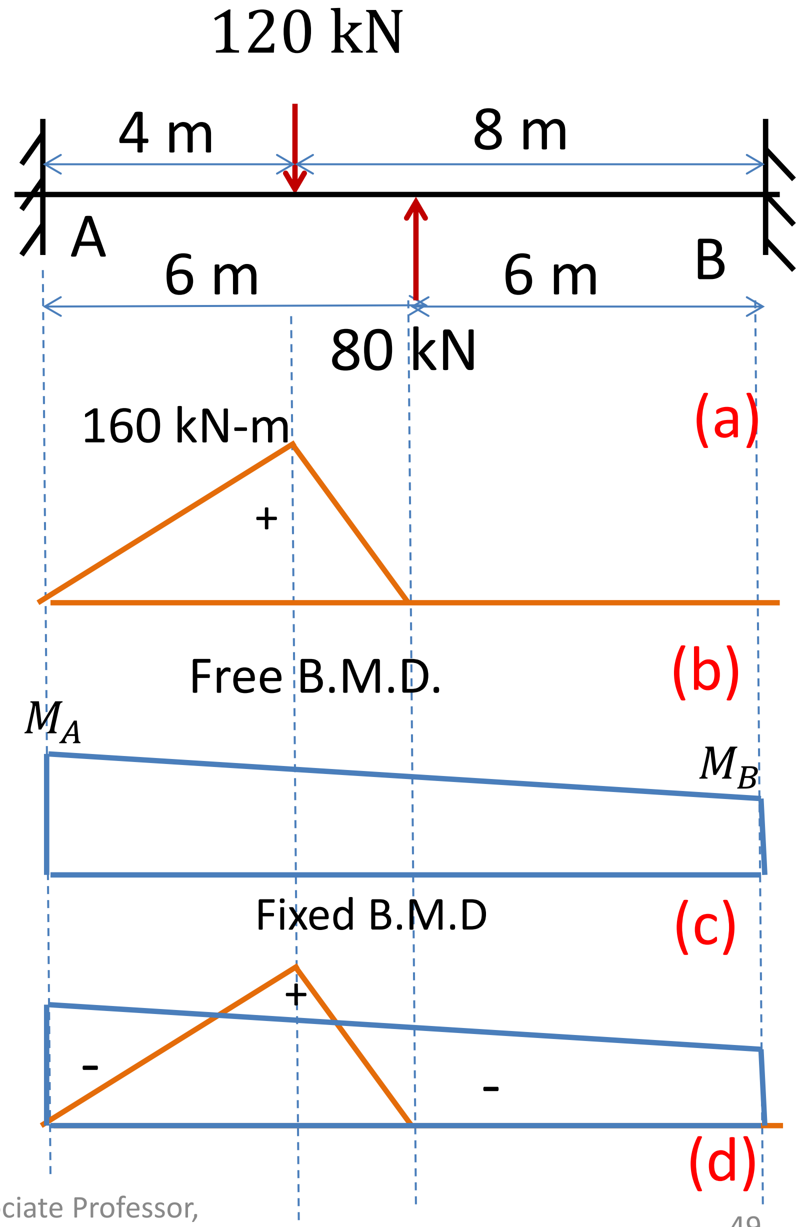
- Solution:
- The M (Free B.M.) and M' (Fixed B.M.) diagrams have been shown in Fig.(b) and (c) respectively.

For the M-Diagram:

$$A = \frac{1}{2} \times 6 \times 160 = 480\ kNm$$

For the M' diagram:

$$A' = \frac{M_A + M_B}{2} \times 12 = 6(M_A + M_B)$$



Fixed Beams - Problems

- Area of the fixed B.M. D. = Area of the free B.M.D.

A' = A

$$6(M_A + M_B) = 480$$

$$M_A + M_B = 80\text{-----------(1)}$$

The distance of the centroid of the free B.M. D. from A = The distance of the centroid of the fixed B.M.D. from A.

i.e., $x = x'$

$$\frac{6+4}{3} = \left(\frac{M_A + 2M_B}{M_A + M_B}\right) \times \frac{12}{3}$$

$$(M_A + 2M_B)12 = (M_A + M_B)10$$

$$12M_A + 24\,M_B - 10\,M_A - 10M_B = 0$$

$$2M_A + 14\,M_B = 0$$

$$M_A = -7M_B \;\; - - - - -(2)$$

Fixed Beams - Problems

- Substitute $M_A = -7M_B$ in equation (1)

$$-7M_B + M_B = 80$$

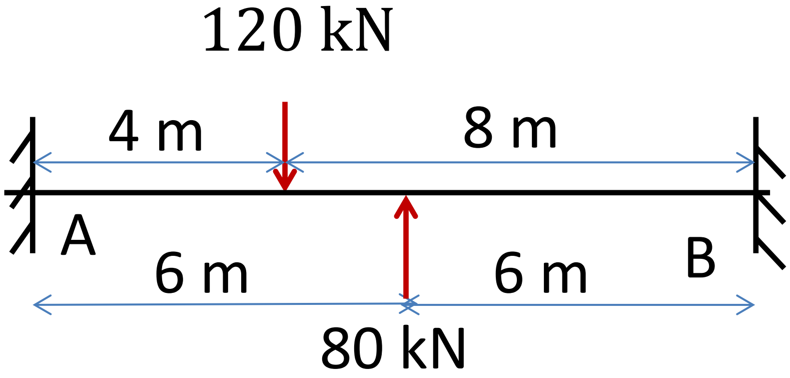
$$\therefore M_B = \frac{-80}{6} = -13.33$$

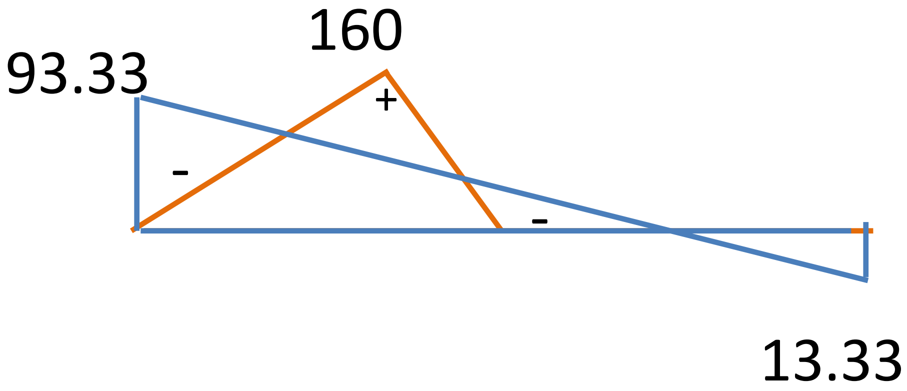
$$M_B = -13.33 \text{ kNm}$$

$$M_A = -7M_B$$

$$= -7(-13.33) = 93.33$$

$$\therefore M_A = 93.33\ kNm$$



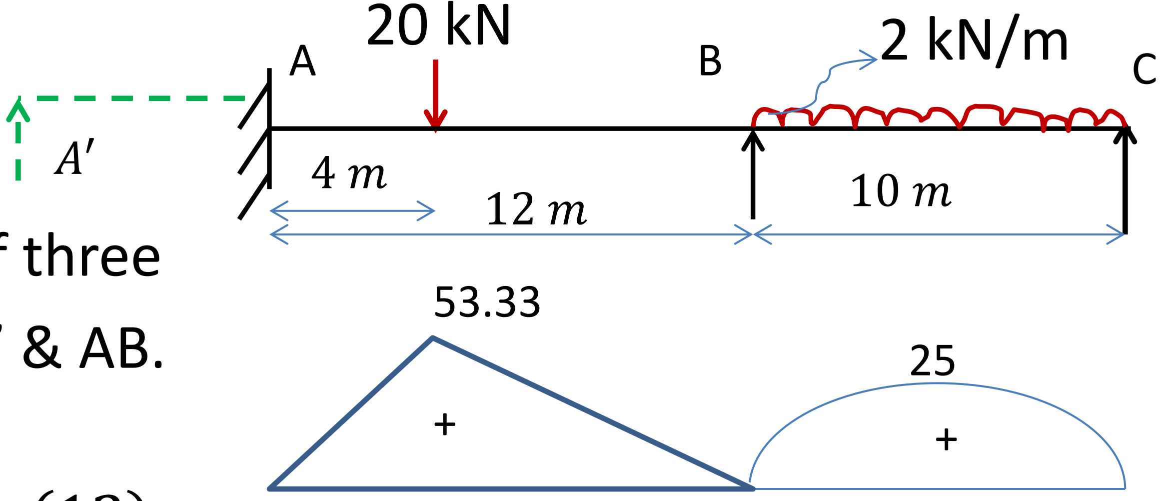


Problems

- A two span continuous beam ABC is fixed at the left end A and placed over simple supports at B and C such that AB=12 m and BC=10 m. It carries a concentrated load of 20 kN at 4 m from the end A. In addition, the beam carries a uniformly distributed load of 2kN/m over BC. Assuming uniform section throughout, analyse the beam and sketch the shear force and B.M.diagrams.

Problems

- Imagine a part A' to the left of A such that AA' = 0.



- Applying theorem of three Moments for spans AA' & AB.

We have ,

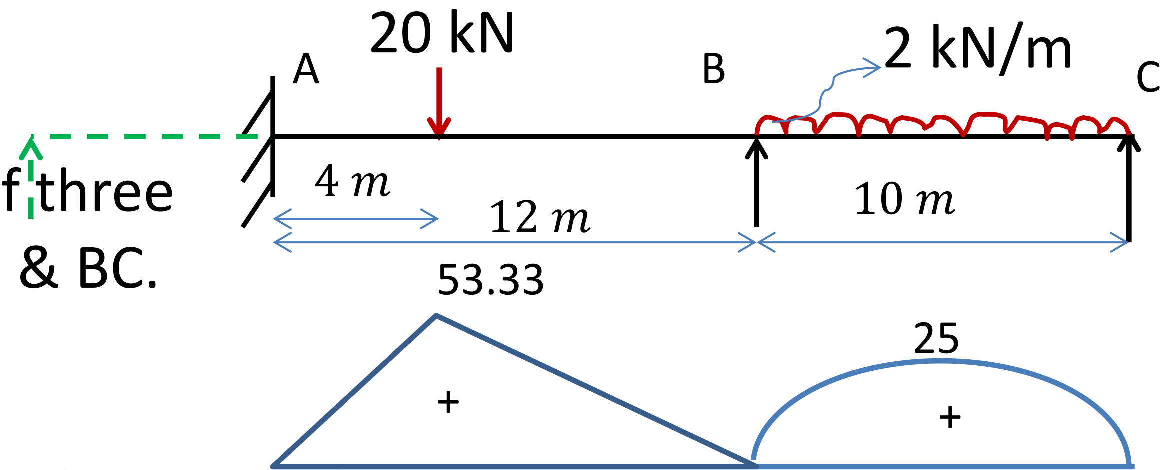
$$0 + 2M_A(0 + 12) + M_B(12)$$

$$= 0 + \frac{6 \times \left(\frac{1}{2} \times 12 \times 53.33 \times 6.67\right)}{12}$$

$$24\ M_A + 12\ M_B = 1067.13$$

$$2M_A + M_B = 88.93\ \ ------(1)$$

Problems



- Applying theorem of three Moments for spans AB & BC.

We have ,

$$M_A(12)+2M_B(12+10)+M_c(12)$$

$$=\frac{6\times\left(\frac{1}{2}\times 12\times 53.33\right)\times 5.33}{12}+\frac{6\times\left(\frac{2}{3}\times 10\times 25\right)\times 5}{10}$$

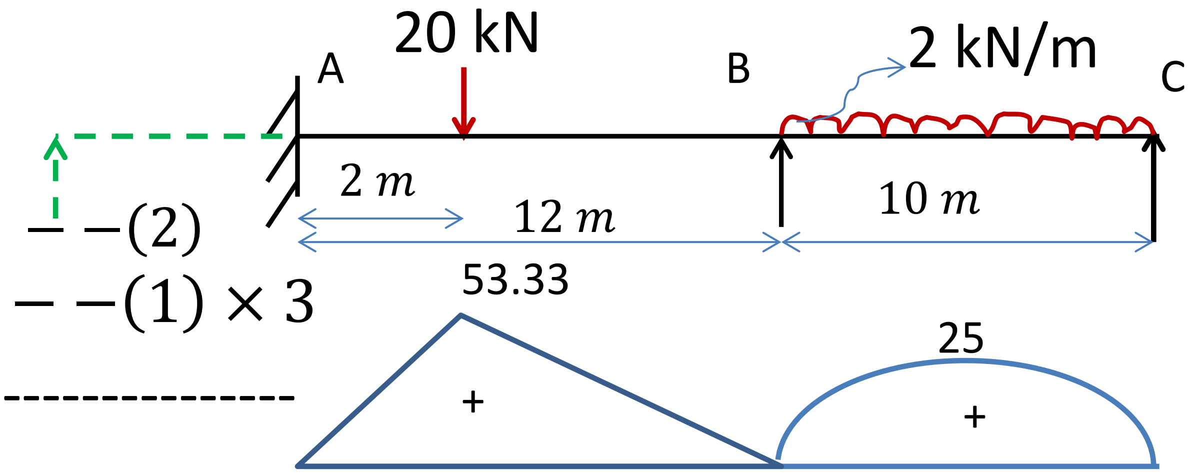
$$12\ M_A+44\ M_B+10M_C=1352.75$$

$$6M_A+22M_B+5M_C=676.375$$

Since at C is simply supported, $M_C=0$

$$\therefore 6M_A+22M_B=676.375---\,-(2)$$

Problems



$$6M_A + 22M_B = 676.375 \;--(2)$$

$$6M_A + 3M_B = 266.78 \;--(1) \times 3$$

$$19M_B = 409.895$$

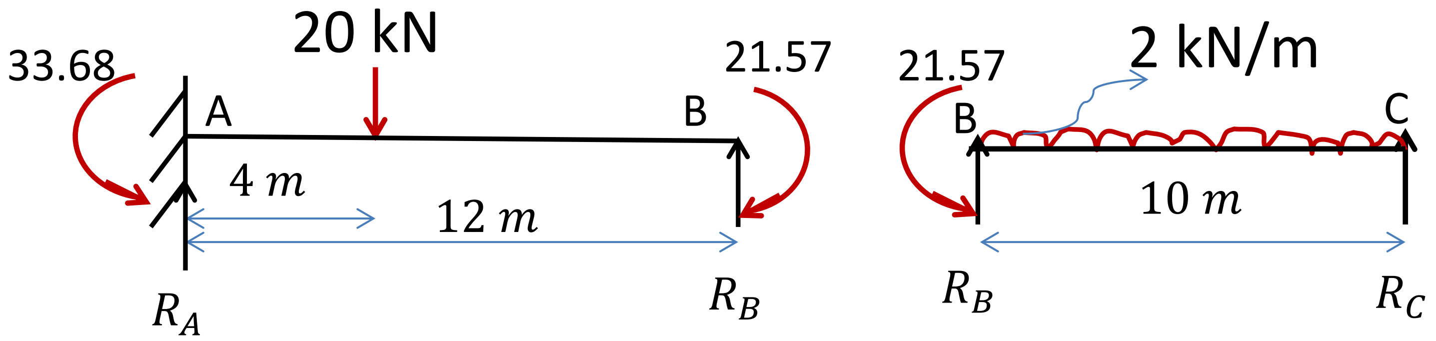
$$\therefore M_B = 21.57 \text{ kNm}$$

By substituting $M_B = 21.57$ in equation (1), we get

$$2M_A + 21.57 = 88.93$$

$$\therefore M_A = 33.68 \text{ kNm}$$

Problems



For span AB, $\sum M_A = 0$

$R_B \times 12 + 33.68 = 21.57 + (20 \times 4)$

$$R_B = \frac{67.875}{12} = 5.66 \text{ kN}$$

$R_A = 20 - 5.66 = 14.34 \text{ kN.}$

For span BC, $\sum M_B$=0

$(R_C\times 10)$+21.57=2× 10 × 5

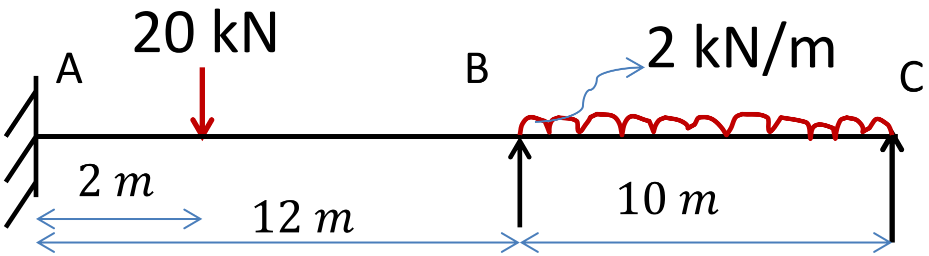
R_C=7.84 kN

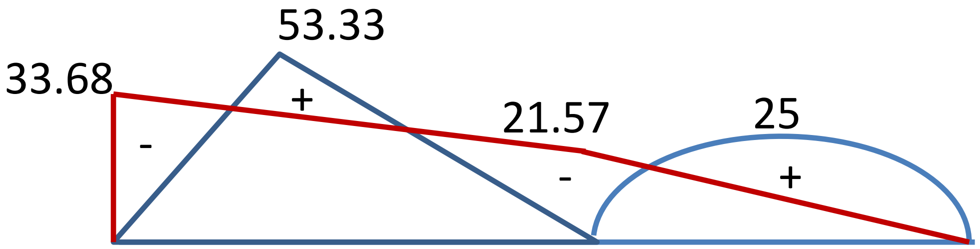
$R_B = 20 - 7.84 = 12.16\text{kN.}$

Problems

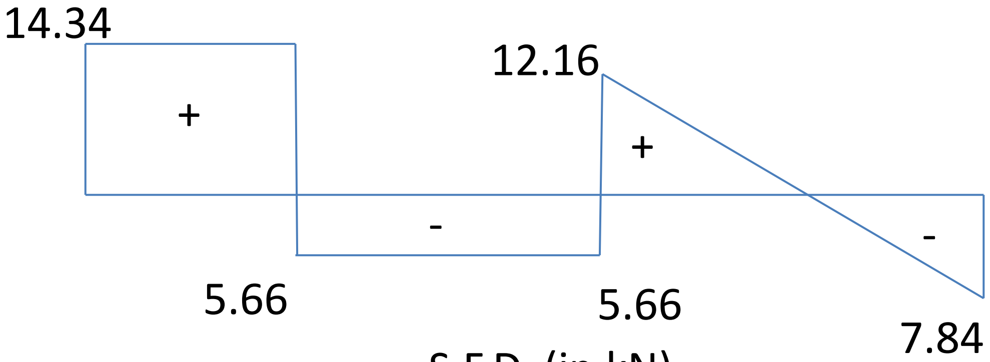
$M_A = 33.68 \text{ kNm}$
$M_B = 21.57 \text{ kNm}$

$R_A = 14.34 \text{ kN}$
$R_B = 5.66\text{kN in span AB}$
$R_B = 12.16\text{kN in span BC}$
R_C=7.84 kN





B.M.D. (in kNm)

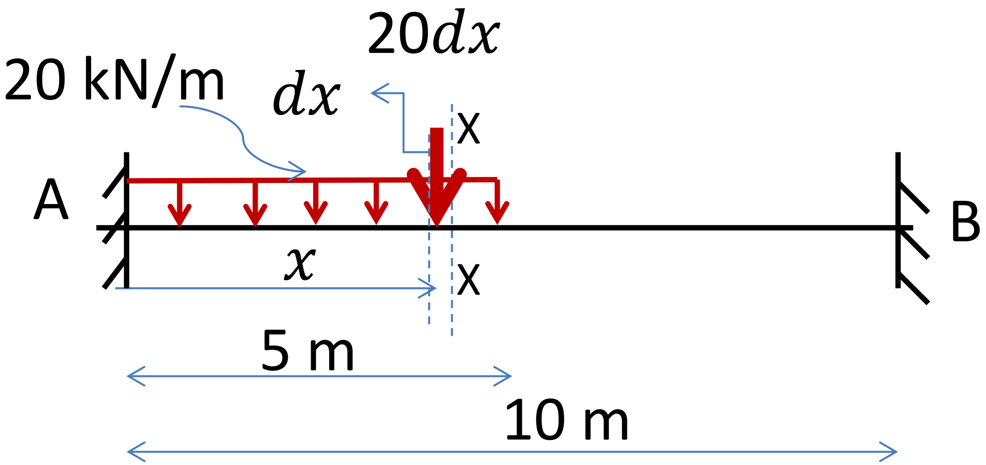


S.F.D. (in kN)

Problems

- A fixed beam of span 10 m carries uniformly distributed load of 20 kN/m over the left half of the span. Analyse the beam and sketch the B.M. and S.F. diagrams for the beam.

Problems



Consider any section XX distant x from the end A, the intensity of loading at XX = 20

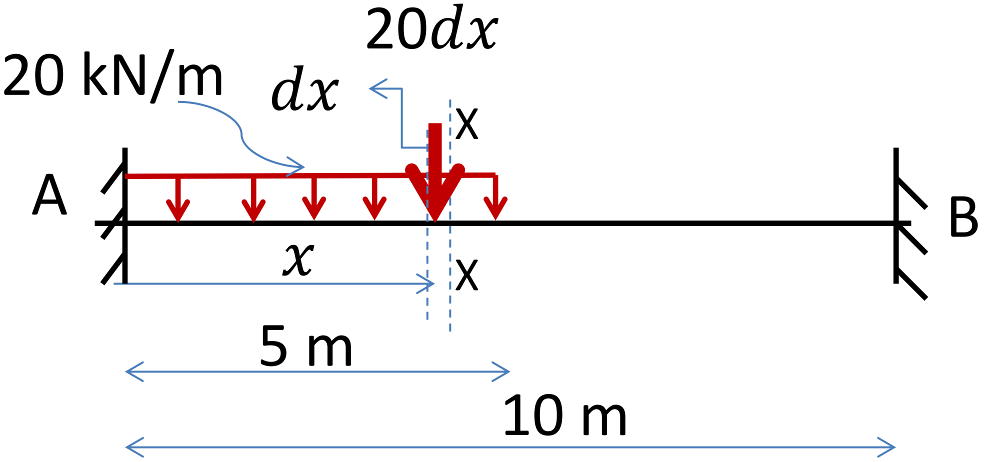
Hence the load acting for an elemental distance $dx = 20dx$

Due to this elemental load the fixed moments are as follows:

$$dM_a = \frac{Wab^2}{L^2} \text{(Formula is derived from first principles)}$$

$$= \frac{20dx \times x \times (10 - x)^2}{10^2} = \frac{20x(100 + x^2 - 20x)dx}{10^2}$$

Problems



and

$$dM_b = \frac{Wba^2}{L^2} \text{(Formula is derived from basic principles)}$$

$$= \frac{20dx \times (6-x) \times (x)^2}{10^2} = \frac{20x^2(6-x)dx}{10^2}$$

Taking fixing moment at A,

$$M_A = \int_0^5 dM_a = \int_0^5 \frac{20}{100} x(100 + x^2 - 20x)dx$$

Problems

$$M_A = \int_0^5 \frac{20}{100} x(100 + x^2 - 20x)dx$$

$$= \frac{20}{100}\left[\frac{100x^2}{2} + \frac{x^4}{4} - \frac{20x^3}{3}\right]\Big|_0^5$$

$$= \frac{20}{100}\left[\frac{100 \times 5^2}{2} + \frac{5^4}{4} - \frac{20 \times 5^3}{3}\right]$$

$$\therefore M_A = 114.58 \text{ kNm}$$

Problems

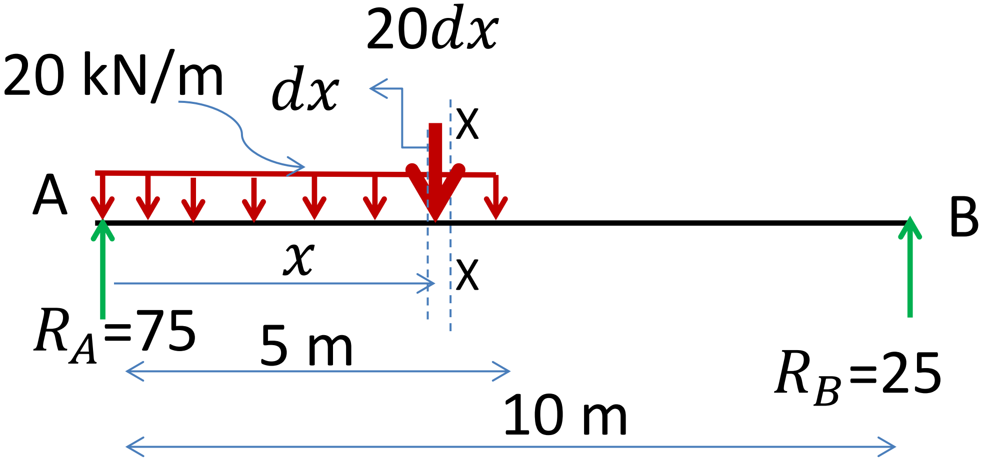
$$M_B = \int_0^5 dM_b = \int_0^5 \frac{20}{10^2} x^2(10 - x)dx$$

$$= \frac{20}{100}\left[\frac{x^3}{3} \times 10 - \frac{x^4}{4}\right]_0^5$$

$$= \frac{20}{100}\left[\frac{5^3 \times 10}{3} - \frac{5^4}{4}\right]$$

$$\therefore M_B = 52.08 \text{ kNm}$$

Problems



Free B.M. calculations:

Taking moments about A,

$R_B \times 10 = 20 \times 5 \times 2.5$

$R_B = 25\ \text{kN}$

$R_A = 20 \times 5 - 25 = 75\ kN$

B.M. is maximum at shear force changes sign, i.e. S.F. is zero.

S.F. at distance x, $F_x = 75 - 20x = 0$

$$\therefore x = \frac{75}{20} = 3.75\ \text{m from A.}$$

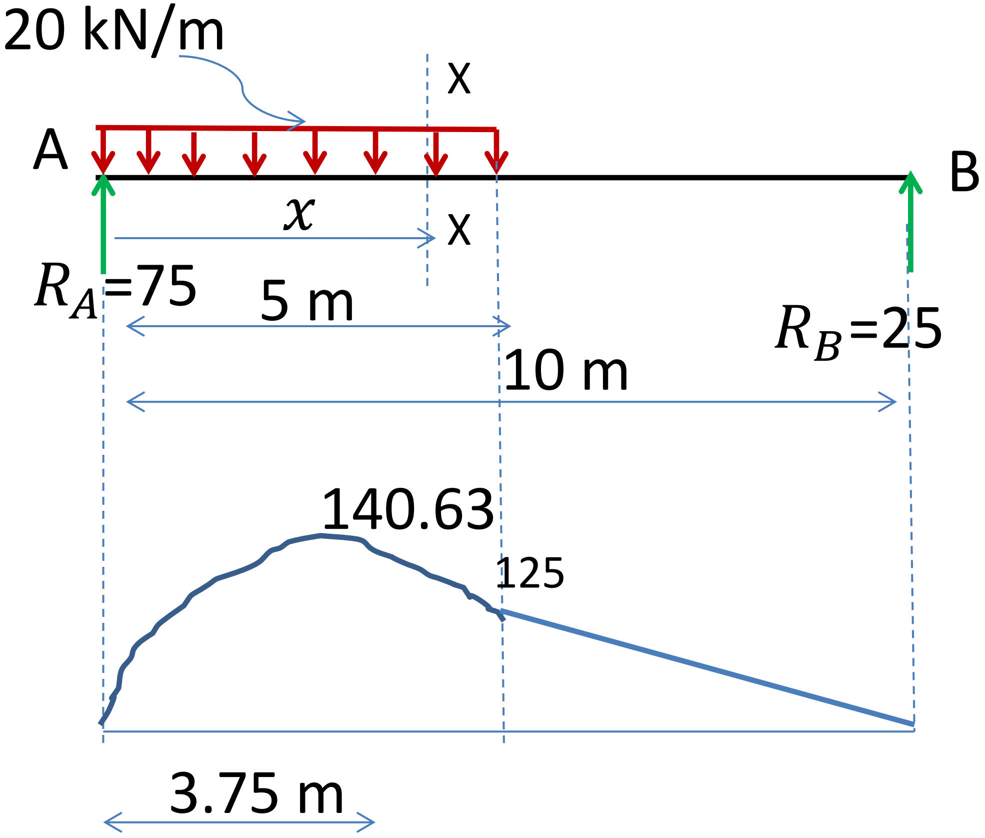
B.M. at $x = M_x = 75x - \frac{20x^2}{2}$

At $x = 5m, M_5 = 75 \times 5 - 20 \times \frac{5^2}{2} = 125\ \text{kNm}$

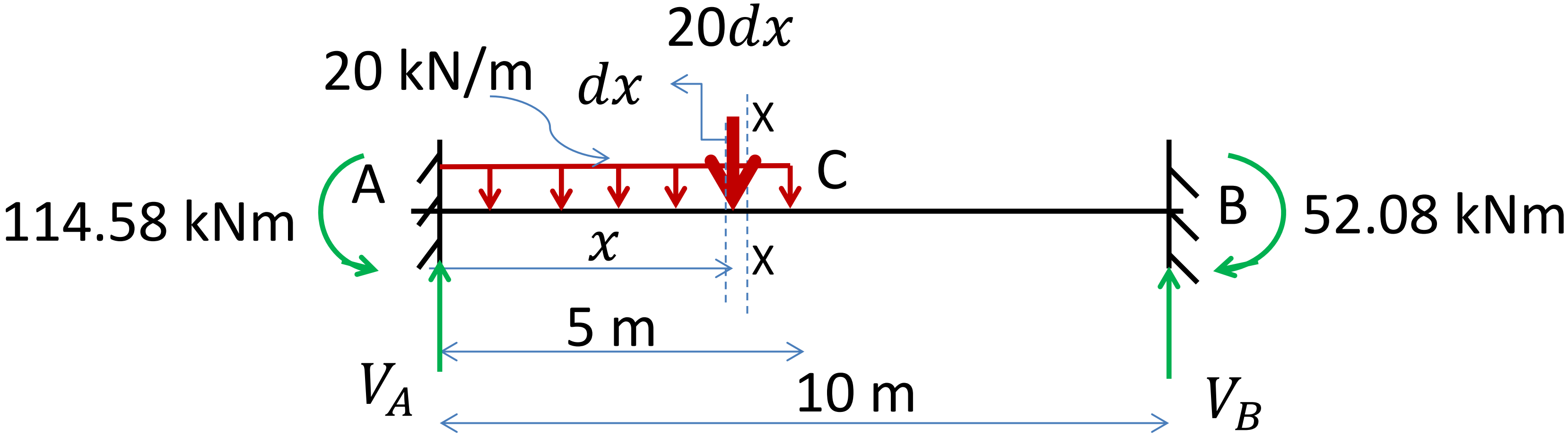
At $x = 3.75\ m, M_{max} = (75 \times 3.75) - \frac{20 \times 3.75^2}{2} = 140.625\ \text{kNm}$

Problems

- M_5=125 kNm
- $M_{3.75} = 140.63 \text{ kNm}$



Problems



SFD Calculations:

To find reactions, Taking moments about B,

$$V_A \times 10 - 114.58 - 20 \times 5 \times (5 + 2.5) + 52.08 = 0$$

$$\therefore V_A = 81.25 \text{ kN}$$

$$V_B = (20 \times 5) - 81.25 = 18.75\ kN$$

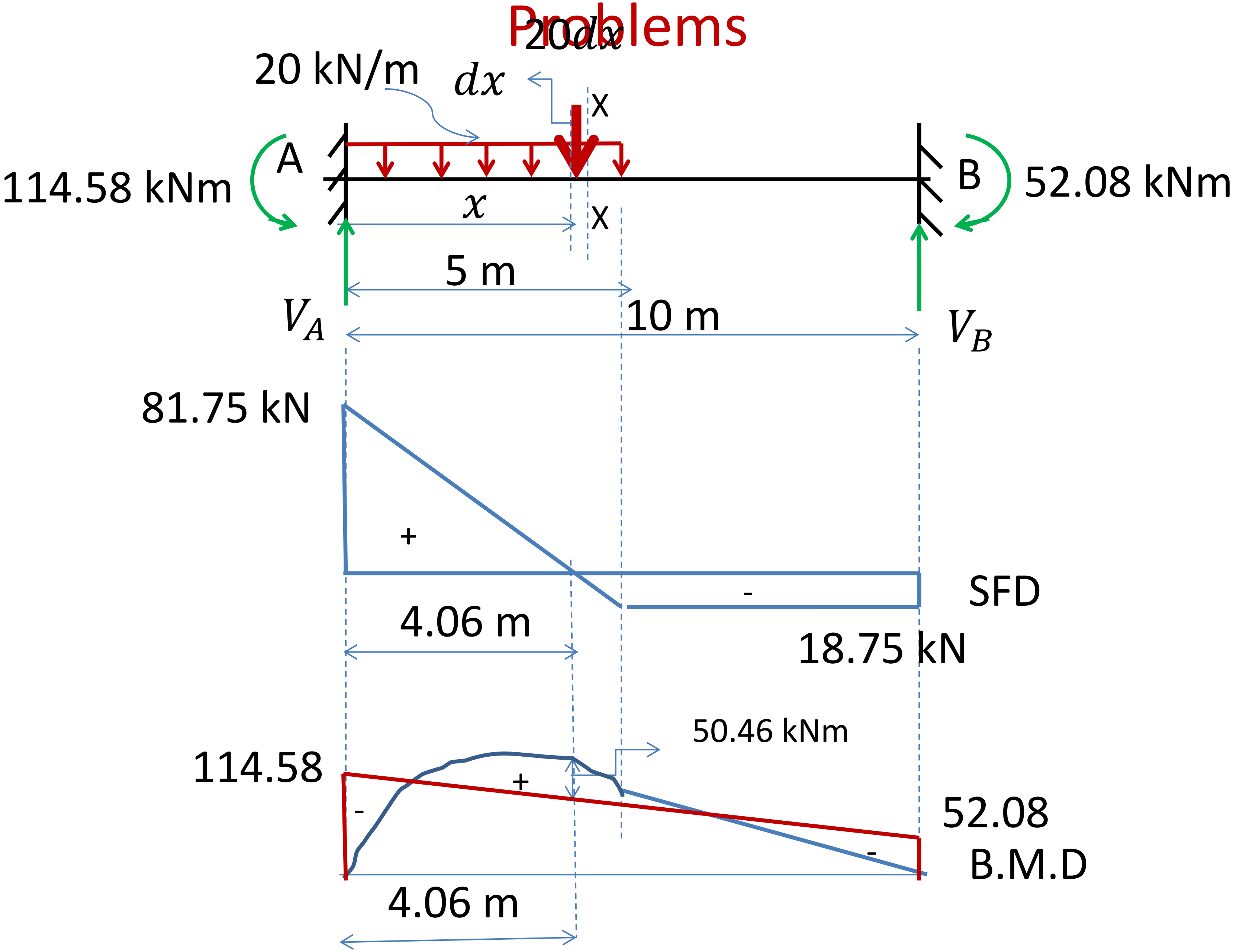
S.F. @A= 81.25 kN

S.F. @ C= 81.25-(20× 5) $= -18.75\ kN$

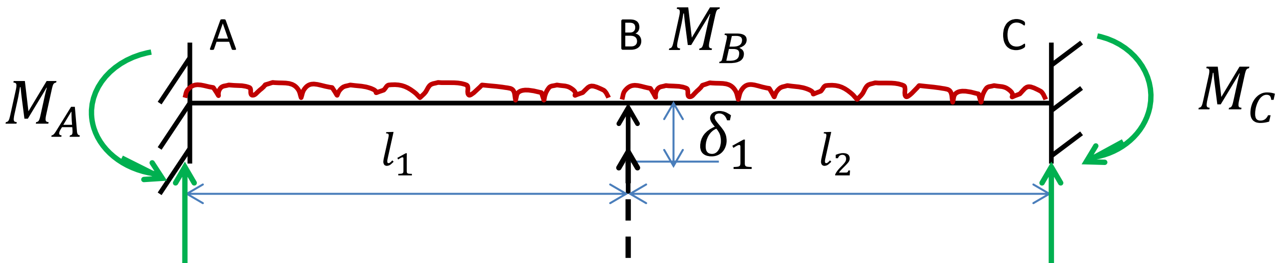
S.F. @B=-18.75 kN

S.F. @ XX=0, 81.25-(20× x) $= 0$, $\therefore x = 4.0625\ m$

Problems


$20dx$
20 kN/m
dx
X
A
114.58 kNm
x
X
B
52.08 kNm
5 m
V_A
10 m
V_B
81.75 kN
+
SFD
-
4.06 m
18.75 kN
50.46 kNm
114.58
+
-
52.08
-
B.M.D
4.06 m

Continuous beam with supports at different levels



- Consider the continuous beam shown in above Figure. Let the support B be δ_1 below A and below C.
- Consider the span AB:
- Let at any section in AB distant x from A the free and fixed bending moments be M_x and $M_x{}'$ respectively.
- Hence the net bending moment at the section is given by

$$EI\frac{d^2y}{dx^2} = M_x - M_x{}'$$

- Multiplying by x, we get

$$EIx\frac{d^2y}{dx^2} = M_x x - M_x{}'x$$

Continuous beam with supports at different levels

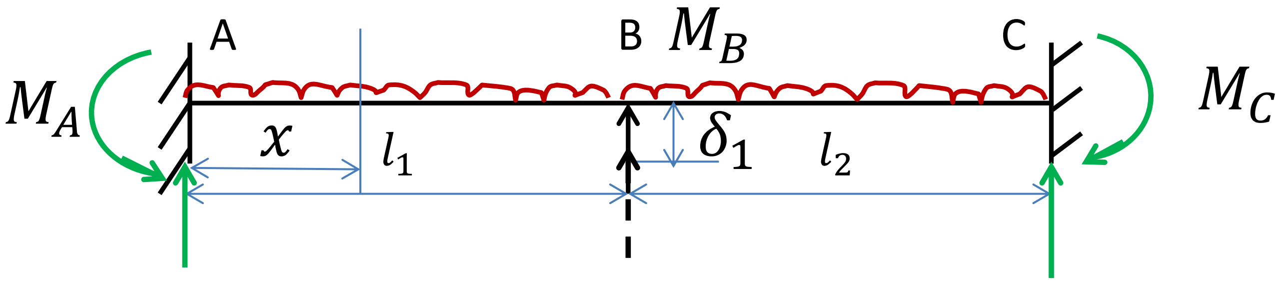
- $EIx\frac{d^2y}{dx^2} = M_x x - {M_x}' x$

- Integrating from $x = 0\ to\ x = l_1$, we get,

$$EI\int_{0}^{l_1} x\frac{d^2y}{dx^2} = \int_{0}^{l_1} M_x x\,dx - \int_{0}^{l_1} {M_x}' x\,dx$$

$$EI\left[x.\frac{dy}{dx} - y\right]_0^{l_1} = \int_{0}^{l_1} M_x x\,dx - \int_{0}^{l_1} {M_x}' x\,dx \qquad ---(1)$$

Continuous beam with supports at different levels



- But it may be such that

  At $x = 0$, deflection $y = 0$

- $At\ x = l_1, y = -\delta_1;\ and\ slope\ at\ B\ for AB, \frac{dy}{dx} = \theta_{BA}$

- $\int_0^{l_1} M_x x\, dx = a_1 \overline{x_1}$ = Moment of the free B. M. D. on AB about A .

- $\int_0^{l_1} M_x{}' x\, dx = a_1'\ \overline{x_1}'$ = Moment of the fixed B. M. D. on AB about A.

Continuous beam with supports at different levels

$$EI\left[x.\frac{dy}{dx}-y\right]_0^{l_1}=\int_0^{l_1}M_x x\,dx-\int_0^{l_1}M_x{}'x\,dx\quad --(1)$$

- Therefore the equation (1) is simplified as,

$$EI\ [l_1\theta_{BA}-(-\delta_1)]\ =a_1\overline{x_1}\ -a_1'\overline{x_1}'.$$

But a_1' = area of the fixed B.M.D. on AB = $\frac{(M_A+M_B)}{2}l_1$

$\overline{x_1}'$ = $\text{Centroid of the fixed B. M. D. from A}$ = $\frac{(M_A+2M_B)}{M_A+M_B}\frac{l_1}{3}$

Continuous beam with supports at different levels

- Therefore,

$$\boldsymbol{a_1^{'}\overline{x_1}^{'}} = \boldsymbol{\frac{(M_A+M_B)}{2}l_1 \times \left(\frac{M_A+2M_B}{M_A+M_B}\right)\frac{l_1}{3} = (M_A+2M_B)\frac{l_1^2}{6}}$$

$$\therefore EI(l_1\theta_{BA}+\delta_1) = a_1\overline{x_1} - (M_A+2M_B)\frac{{l_1}^2}{6}$$

$$6EI\ \theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - \frac{6EI\delta_1}{l_1} - (M_A+2M_B)l_1 --- (2)$$

Similarly by considering the span BC and taking C as origin it can be shown that,

$$6EI\ \theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - \frac{6EI\delta_2}{l_2} - (M_C+2M_B)l_2 --- (3)$$

$$\theta_{BC} = \text{slope for span CB at B}$$

Continuous beam with supports at different levels

- But $\theta_{BA} = -\theta_{BC}$ as the direction of x from A for the span AB, and from C for the span CB are in opposite direction.
- And hence, $\theta_{BA} + \theta_{BC} = 0$

$$6EI\ \theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - \frac{6EI\delta_1}{l_1} - (M_A + 2M_B)l_1 ---(2)$$

$$6EI\ \theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - \frac{6EI\delta_2}{l_2} - (M_C + 2M_B)l_2 ---(3)$$

Adding equations (2) and (3), we get

$$6EI\ (\theta_{BA}+\theta_{BC}) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - \frac{6EI\delta_1}{l_1} - \frac{6EI\delta_2}{l_2} - [M_A l_1 + 2M_B(l_1 + l_2) + M_c l_2]$$

Continuous beam with supports at different levels

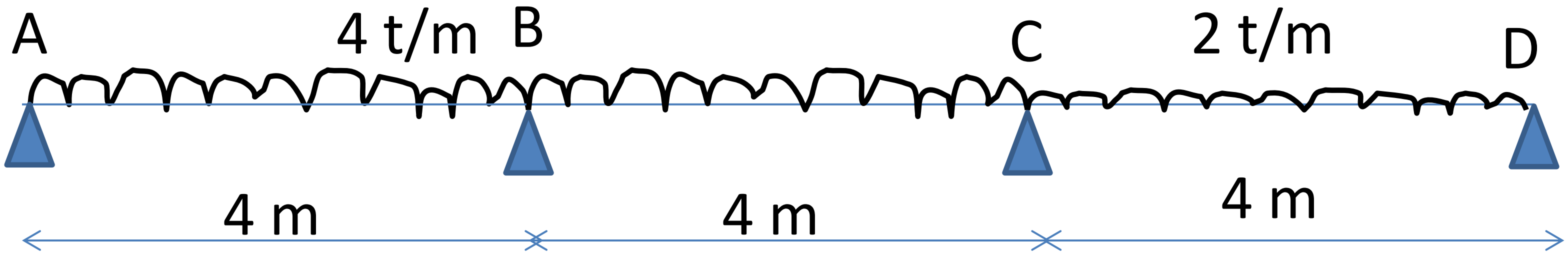
$$6EI\,(\theta_{BA}+\theta_{BC}) = \frac{6a_1\overline{x_1}}{l_1}+\frac{6a_2\overline{x_2}}{l_2}-\frac{6EI\delta_1}{l_1}-\frac{6EI\delta_2}{l_2} - [M_A l_1 + 2M_B(l_1+l_2)+M_c l_2]$$

$$0 = \frac{6a_1\overline{x_1}}{l_1}+\frac{6a_2\overline{x_2}}{l_2}-\frac{6EI\delta_1}{l_1}-\frac{6EI\delta_2}{l_2} - [M_A l_1 + 2M_B(l_1+l_2)+M_c l_2]$$

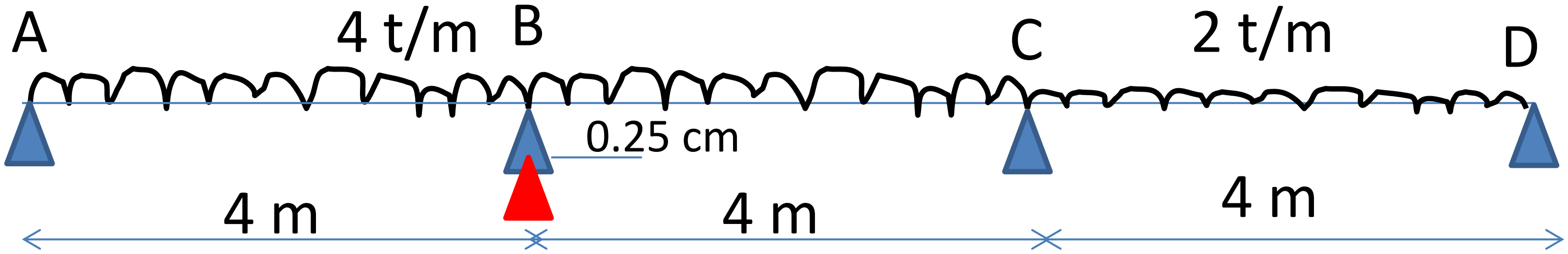
$$[M_A l_1 + 2M_B(l_1+l_2)+M_c l_2] = \frac{6a_1\overline{x_1}}{l_1}+\frac{6a_2\overline{x_2}}{l_2} - 6EI\left(\frac{\delta_1}{l_1}+\frac{\delta_2}{l_2}\right)$$

Problems

- The following Figure shows a continuous beam carrying an external loading. If the support B sinks by 0.25 cm below the level of the other supports find support moments. Take I for section= 15000 cm^4 and E=2x10^3 t/cm^2.



Problems

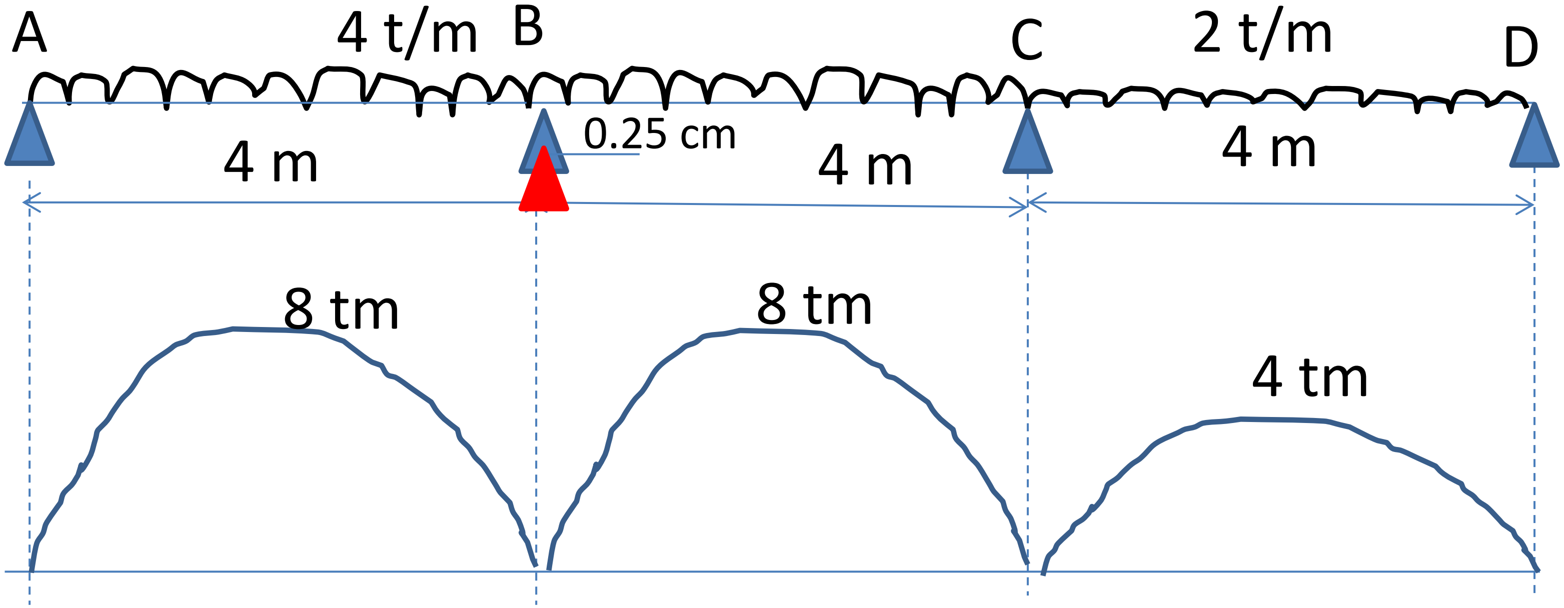


- The theorem of three moments for two spans AB and BC is as follows,

$$[M_A l_1 + 2M_B(l_1 + l_2) + M_c l_2] = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

- Consider the spans AB and BC,
- $M_A = 0$
- $\delta_1 = +0.25\ \text{cm}$
- $\delta_2 = +0.25\ \text{cm}$

Problems

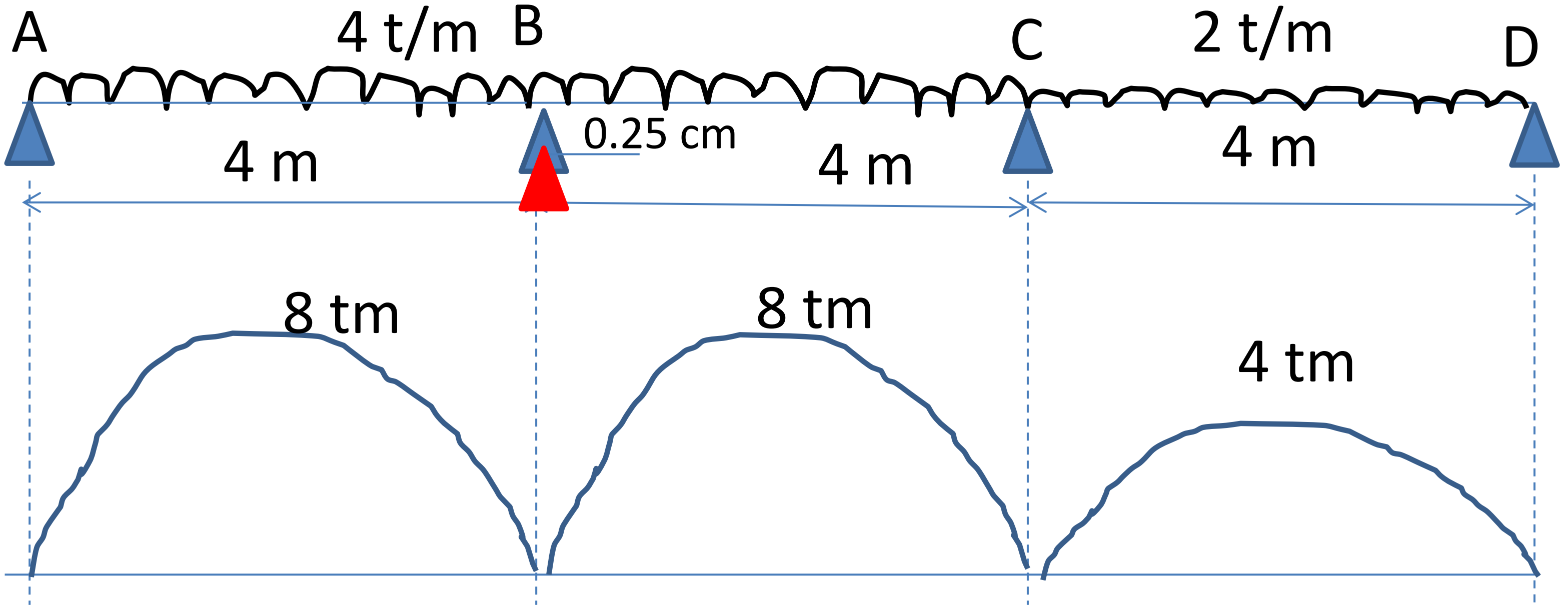


Free BMD

- $a_1 = \frac{2}{3} \times 4 \times 8$
- $\frac{6a_1\overline{x_1}}{l_1} = \frac{6\times\frac{2}{3}\times4\times8\times2}{4} = 64$
- $\frac{6a_2\overline{x_2}}{l_2} = \frac{6\times\frac{2}{3}\times4\times8\times2}{4} = 64$

$$6EI = \frac{6 \times 2 \times 10^3 \times 15000}{100^2}$$
$$= 18000\ tm^2$$

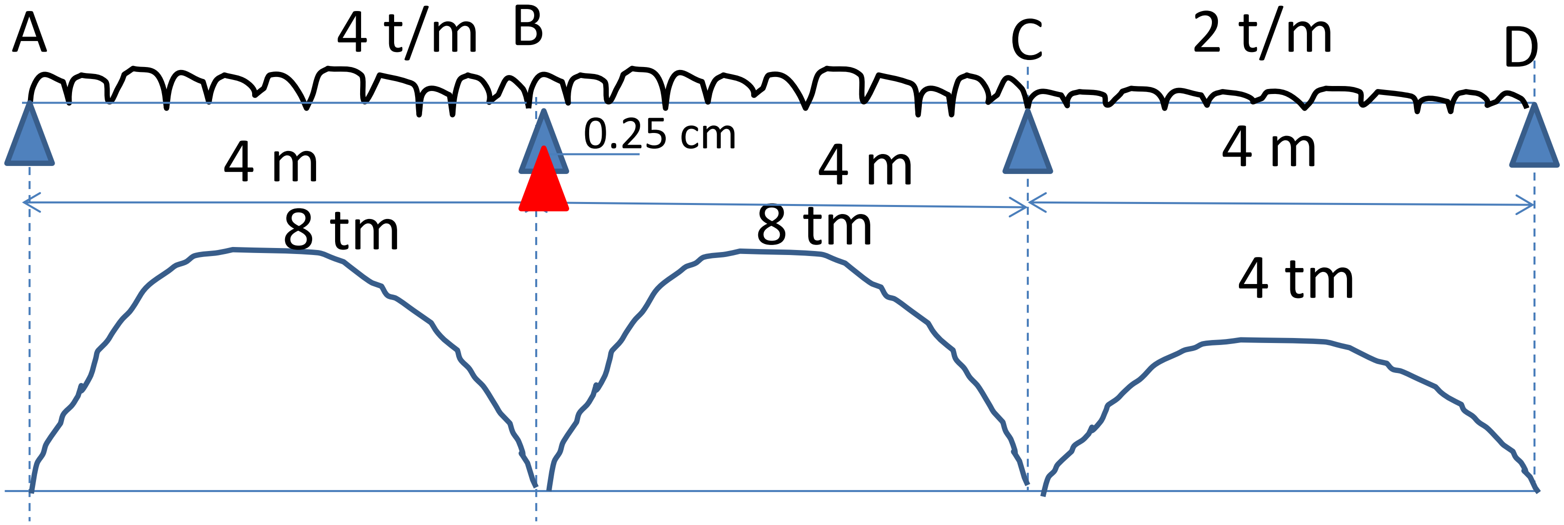
Problems



$$\left[M_A l_1 + 2M_B(l_1 + l_2) + M_c l_2\right] = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

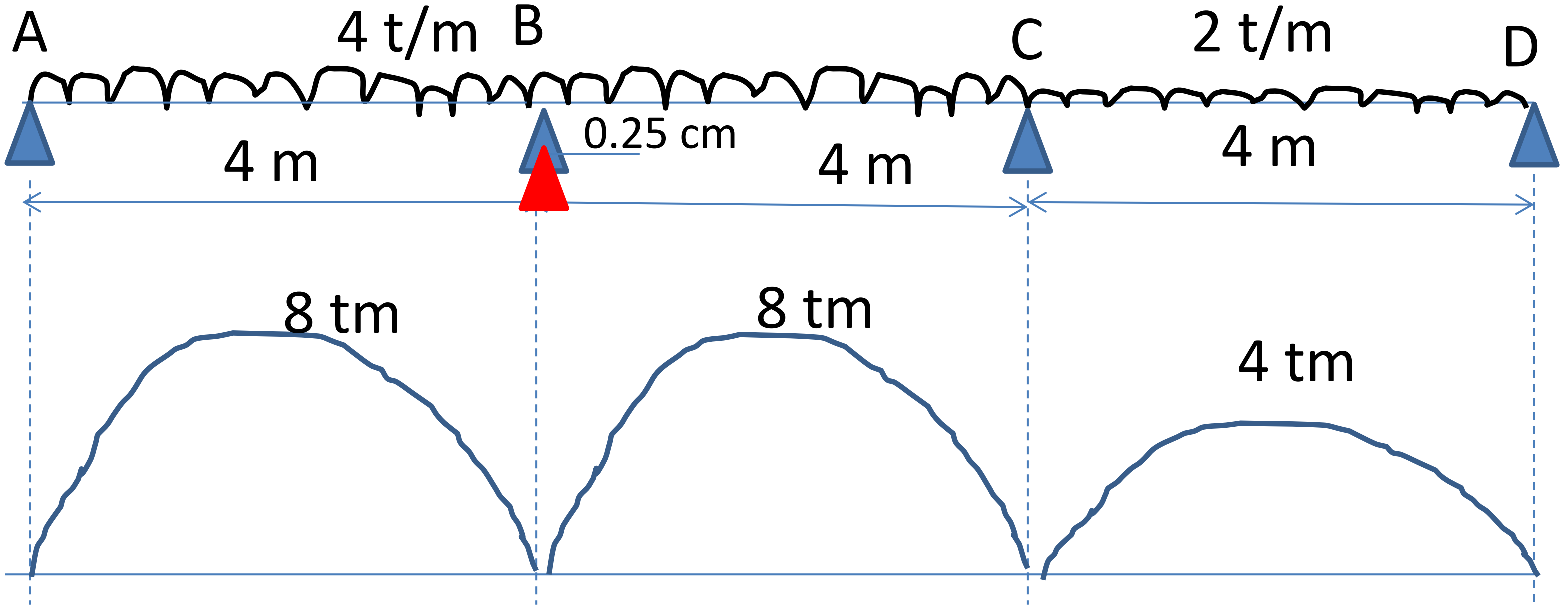
- $\therefore 0 + 2M_B(4+4) + 4M_C = 64 + 64 - 18000\left(\frac{0.25}{400} + \frac{0.25}{400}\right)$
- $\therefore 16M_B + 4M_C = 128 - 22.5$
- $16M_B + 4M_C = 105.5$
- $4M_B + M_C = 26.375 \;----(1)$

Problems



- Now consider the spans BC and CD,
- $M_d = 0,$
- $\delta_1 = -0.25$ cm
- $\delta_2 = 0$

Problems



$$[M_A l_1 + 2M_B(l_1 + l_2) + M_c l_2] = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

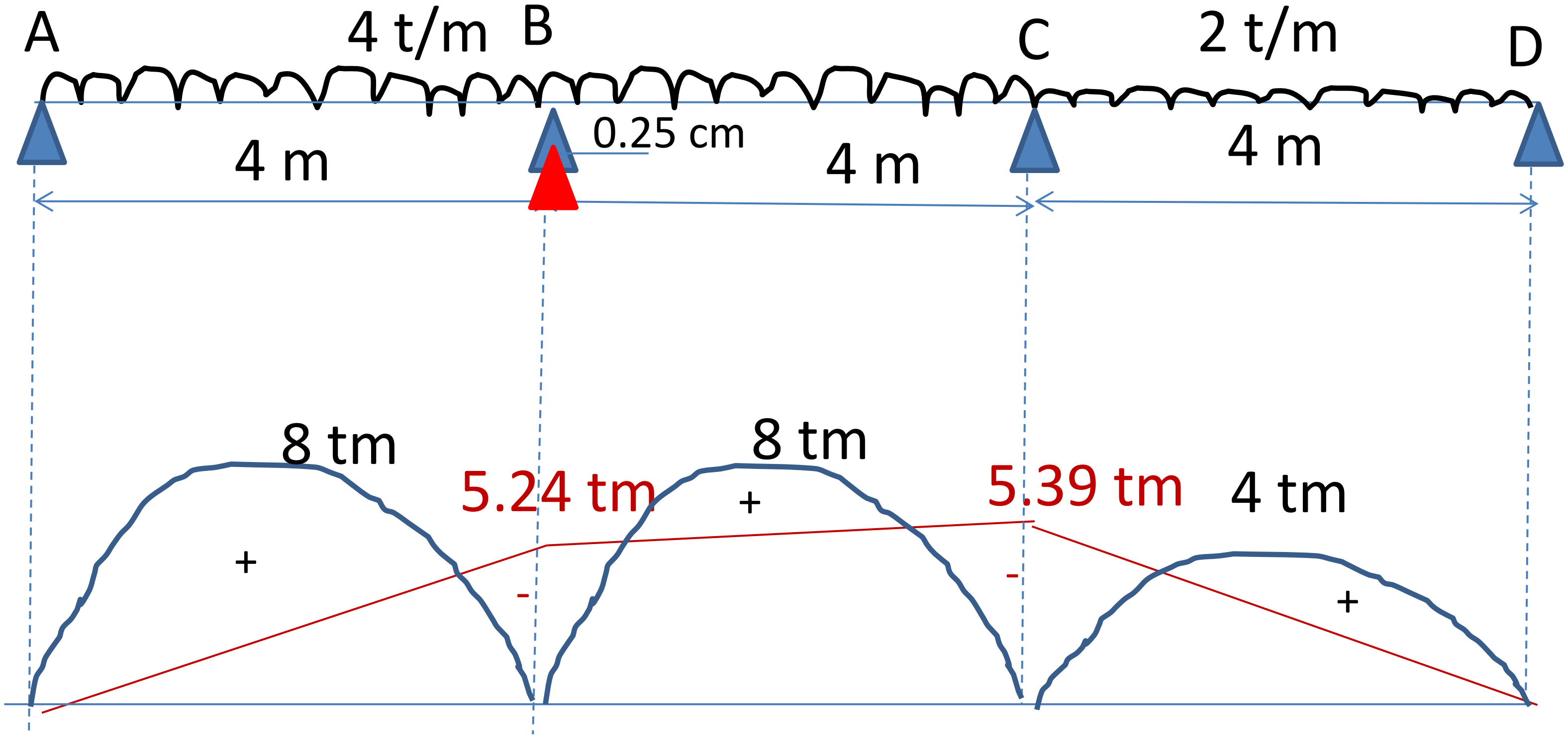
$$M_B \times 4 + 2M_C(4 + 4) + 0 = 64 + 32 - 18000\left(\frac{-0.25}{400} + \frac{0}{400}\right)$$

- $\therefore 4M_B + 16M_C = 96 + 11.25$
- $4M_B + 16M_C = 107.25 ---- -(2)$

Problems

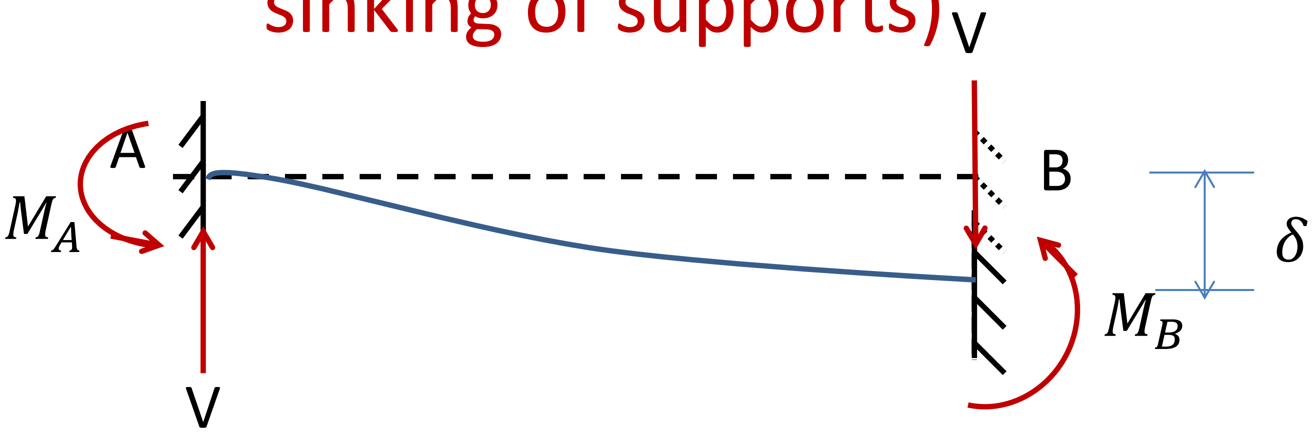
- $4M_B + M_C = 26.375 \; ---- -(1)$
- $4M_B + 16M_C = 107.25 ---- -(2)$
- Solving (1) and (2), we get,
- $M_B = 5.24 \text{ tm (hogging)}.$
- $M_C = 5.39 \text{ tm (hogging)}.$

Problems



BMD

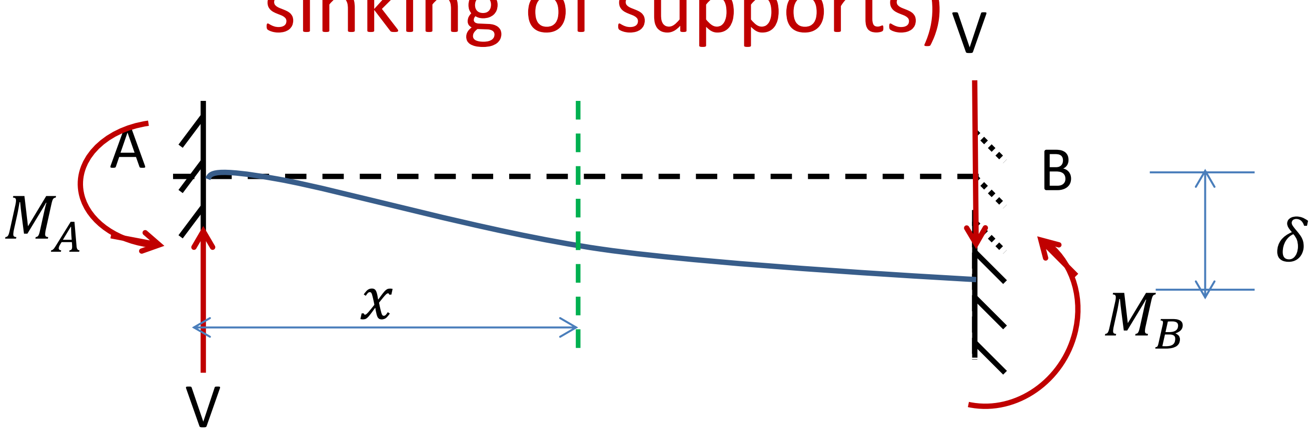
Fixed beam with ends at different levels (Effect of sinking of supports)



M_A is negative (hogging) and M_B is positive (sagging). Numerically M_A and M_B are equal.

Let V be the reaction at each support.

Fixed beam with ends at different levels (Effect of sinking of supports)



Consider any section distance x from the end A.

Since the rate of loading is zero, we have, with the usual notations

$$EI\frac{d^4y}{dx^4} = 0$$

Integrating, we get,

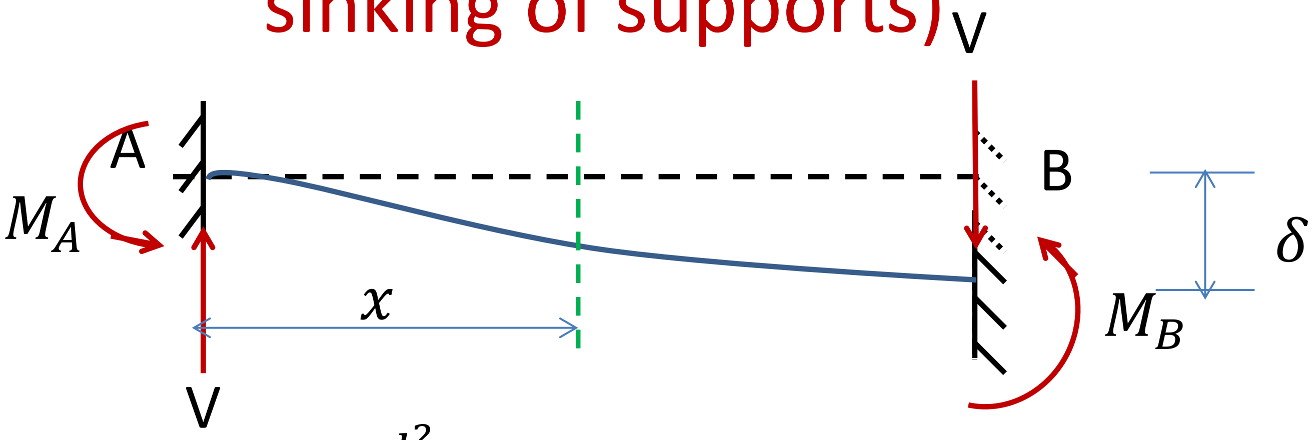
Shear force = $EI\frac{d^3y}{dx^3} = C_1$

Where C_1 is a constant

At $x = 0$, $\quad S.F. = +\mathrm{V}$

$$\therefore C_1 = \mathrm{V}$$

Fixed beam with ends at different levels (Effect of sinking of supports)



B.M. at any section = $EI\dfrac{d^2y}{dx^2} = Vx + C_1$

At $x = 0,\ B.M. = -M_A$

$$\therefore\ C_2 = -M_A$$

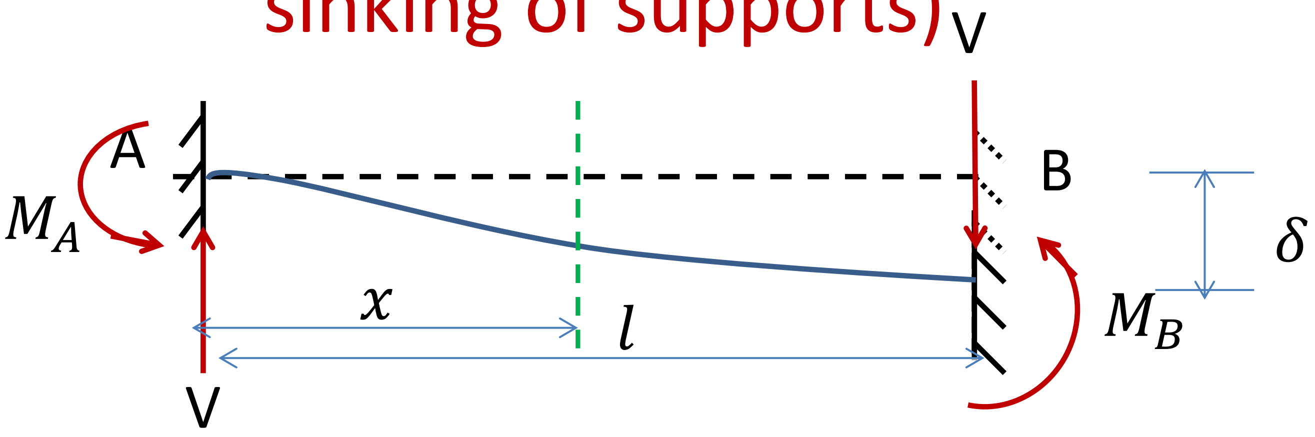
$$\therefore EI\frac{d^2y}{dx^2} = Vx - M_A$$

Integrating again,

$EI\dfrac{dy}{dx} = \dfrac{V}{2}x^2 - M_Ax + C_3$ (Slope equation)

But at $x = 0,\ \dfrac{dy}{dx} = 0 \qquad \therefore C_3 = 0$

Fixed beam with ends at different levels (Effect of sinking of supports)



Integrating again,

$$EI\ y = \frac{Vx^3}{6} - \frac{M_A x^2}{2} + C_4 \quad \text{------ (Deflection equation)}$$

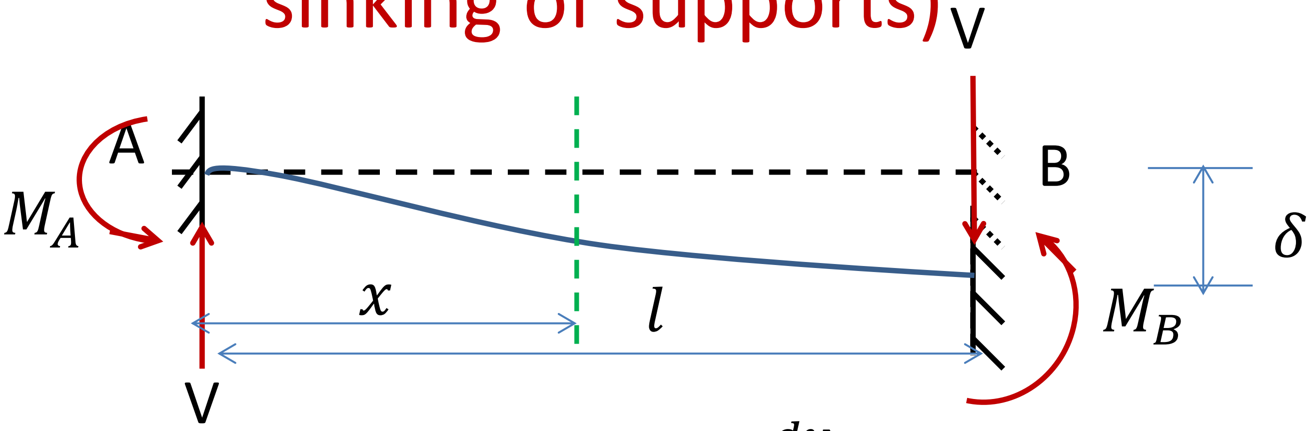
But at $x = 0, y = 0$

$$\therefore C_4 = 0$$

At $x = l, y = -\delta$

$$-EI\ \delta = \frac{Vl^3}{6} - \frac{M_A l^2}{2} ---------- -(\mathrm{i})$$

Fixed beam with ends at different levels (Effect of sinking of supports)



But we also know that at B, $x = l \; and \; \frac{dy}{dx} = 0$

And substitute in slope Eq. $EI\frac{dy}{dx} = \frac{V}{2}x^2 - M_A x$

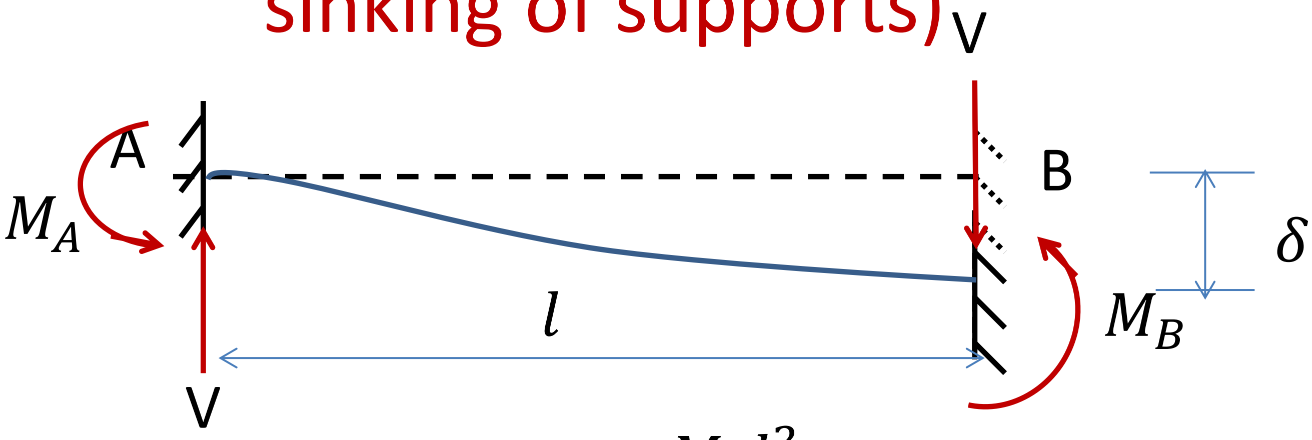
$$\therefore 0 = \frac{Vl^2}{2} - M_A l$$

$$\therefore V = \frac{2M_A}{l} --------- -(\text{ii})$$

Substituting in deflection Eq.(i) i.e., $-EI\,\delta = \frac{Vl^3}{6} - \frac{M_A l^2}{2}$;we have,

$$-EI\,\delta = \frac{2M_A}{l} \times \frac{l^3}{6} - \frac{M_A l^2}{2}$$

Fixed beam with ends at different levels (Effect of sinking of supports)



$$EI\ \delta = \frac{M_A l^2}{6}$$

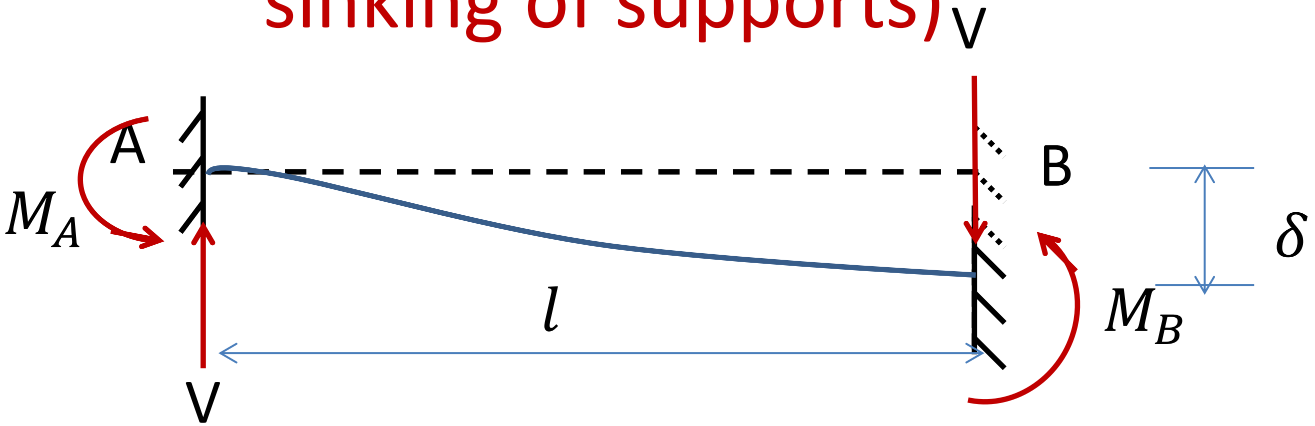
$$\therefore M_A = \frac{6EI\delta}{l^2}$$

Hence the law for the bending moment at any section distant x from A is given by,

$$M = EI\frac{d^2y}{dx^2} = \mathrm{V}x - M_A$$

$$\therefore M = \frac{2M_A}{l}x - \frac{6EI\delta}{l^2}$$

Fixed beam with ends at different levels (Effect of sinking of supports)



$$\text{But for B. M. at B, put x} = \text{l,}$$

$$\therefore M_B = \frac{2M_A}{l} \times l - \frac{6EI\delta}{l^2} = \frac{12EI\delta}{l^2} - \frac{6EI\delta}{l^2} = \frac{6EI\delta}{l^2}$$

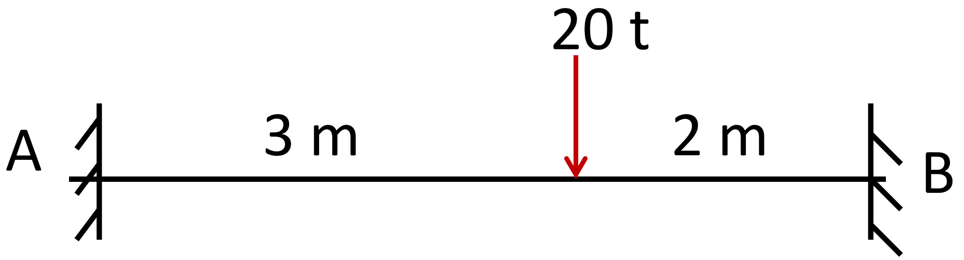
Hence when the ends of a fixed beam are at different levels,

The fixing moment at each end = $\frac{6EI\delta}{l^2}$ numerically.

At the higher end this moment is a hogging moment and at the lower end this moment is a sagging moment.

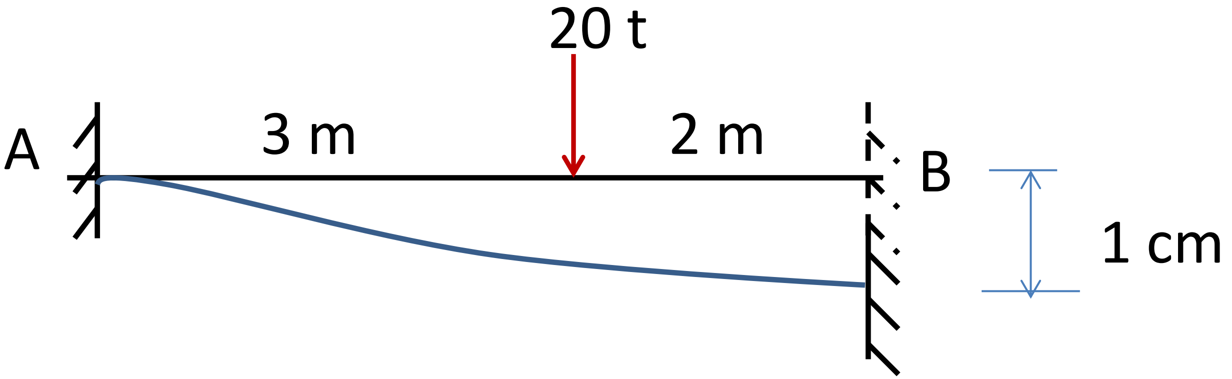
Problems

- A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 meters from the left end. If the right end sinks by 1 cm, find the fixing moments at the supports. For the beam section take I=30,000 cm^4 and E=2x10^3 t/cm^2. Find also the reaction at the supports.



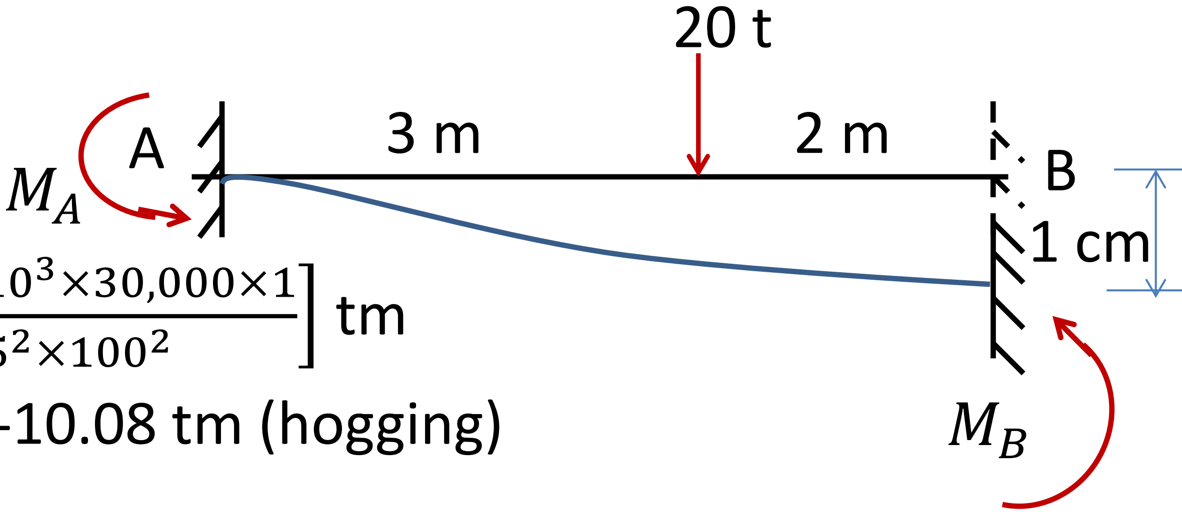
Problems

- A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 meters from the left end.



- The right end sinks by 1 cm, find the fixing moments at the supports.

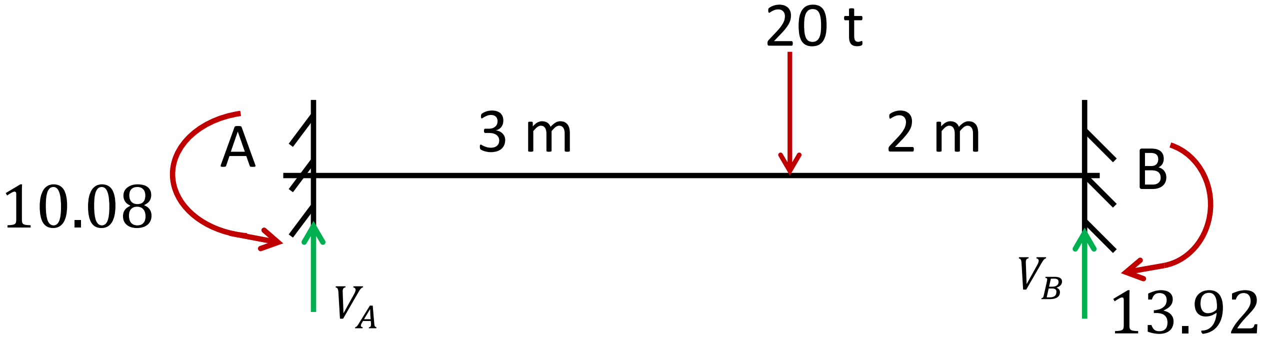
Problems



- $M_A = -\frac{\mathrm{W}ab^2}{l^2} - \frac{6EI\delta}{l^2}$
- $= -\left[\frac{20\times3\times2^2}{5^2} + \frac{6\times2\times10^3\times30{,}000\times1}{5^2\times100^2}\right]$ tm
- $= -[9.6 + 0.48]$ tm=-10.08 tm (hogging)

- $M_B = -\frac{\mathrm{W}ba^2}{l^2} + \frac{6EI\delta}{l^2}$
- $= \left[-\frac{20\times2\times3^2}{5^2} + \frac{6\times2\times10^3\times30{,}000\times1}{5^2\times100^2}\right]$ tm
- $= [-14.4 + 0.48]$ tm= -13.92 tm (hogging)

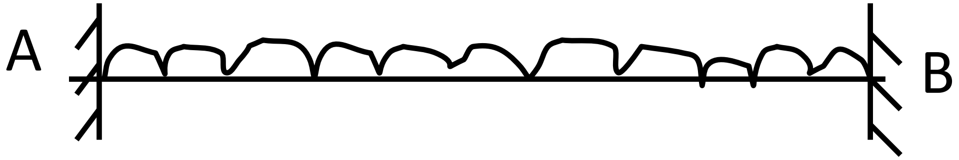
Problems



- <u>Reaction at A:</u>
- $\sum M_B$=0,
- $V_A \times 5 + 13.92 - 10.08 - (20 \times 2) = 0$
- $\therefore V_A = 7.232$ t
- <u>Reaction at B:</u>
- $\therefore V_B = 20 - 7.232 = 12.768$ t.

A.U. question paper problems

- Find the fixed end moments of a fixed beam subjected to u.d.l for whole span of the beam.



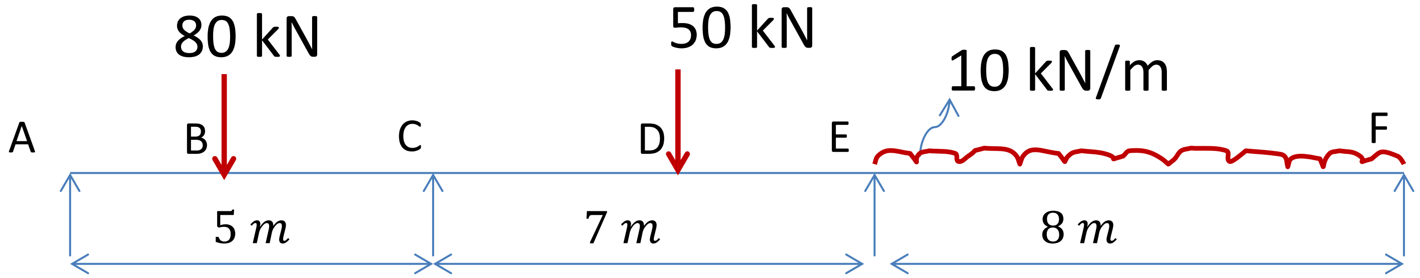
$$M_A = M_B = \frac{Wl^2}{12}$$

A.U. question paper problems

- A fixed beam ACB of span 6 m is carrying a concentrated clockwise couple of 150 kN-m applied at a section 4m from the left end. Find the end moments from the first principles. Draw BM and SF diagrams.(Apr/May2015)

A.U. question paper problems

- For the beam given in Fig., find the moment and reaction at the supports. Draw SFD and BMD. Take AB=2 m; BC=3 m; CD=4 m; DE=3 m; EF=8 m. (Apr/May2015)



A.U. question paper problems

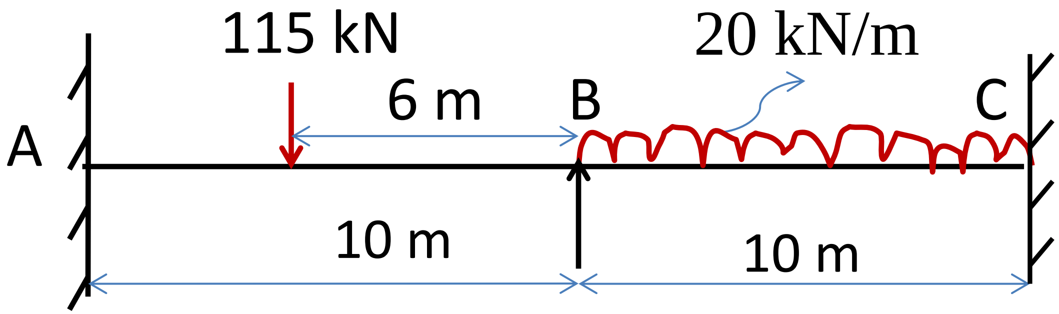
- A continuous beam ABC of uniform section, with span AB and BC as 6m each, is fixed at A and supported at B and C. Span AB carries UDL of 2 kN/m and BC having a midpoint of 12 kN. Find the support moments and the reactions. Also draw the SFD and BMD of the beam. (Nov/Dec 2014)

A.U. question paper problems

- What is the Clapeyron’s theorem of three moments? Derive an expression for Clapeyron’s theorem for three moments. (Nov/Dec 2014)

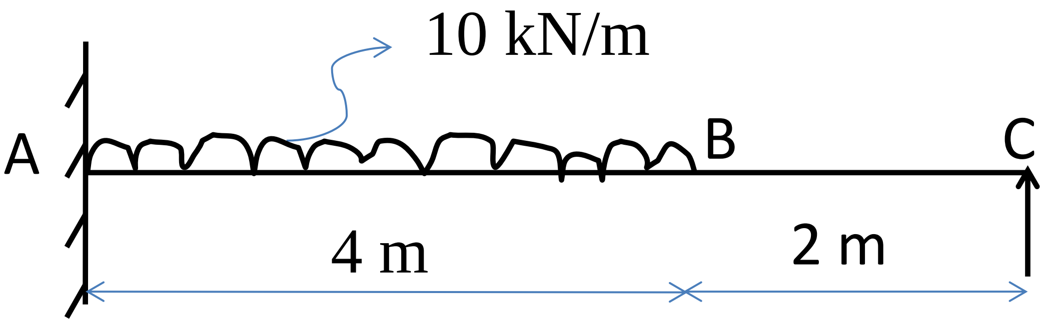
A.U. Question paper problems

- A continuous beam ABC, is loaded as shown in Figure. Find the support moments using three moment equation. Draw shear force and bending moment diagram. (Nov/Dec 2012)



A.U. Question paper problems

- A cantilever ABC is fixed at A and propped at C is loaded as shown in the following Figure. Find the reaction at C. (May/June 2013)



A.U. Question paper problems

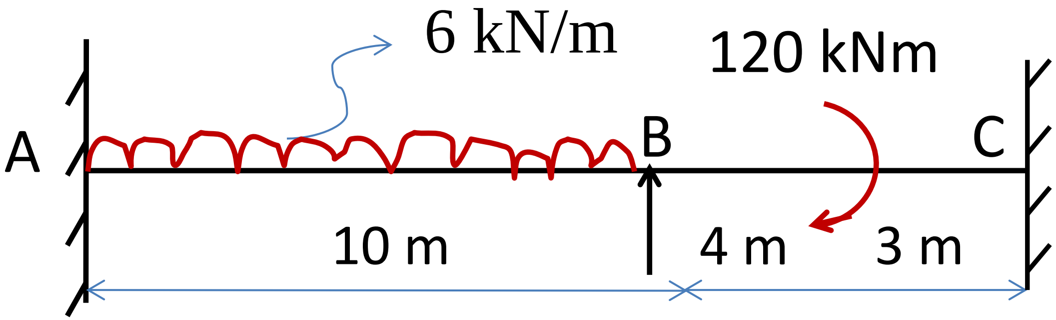
- A cantilever beam ABC of span 6m fixed at A and propped at C is loaded with an udl of 10 kN/m for a length of 4m from the fixed end. Find the prop reaction. Draw shear force and Bending moment diagram. Find the maximum sagging bending moment and point of contraflexure. (Nov/Dec 2012)
-

A.U. Question paper problems

- A propped cantilever of span 6 m is subjected to a u.d.l. of 2kN/m over a length of 4 m from the fixed end. Determine the prop reaction and draw the shear force and bending moment diagrams. (May/June 2012)
-

A.U. Question paper problems

- A two span continuous beam fixed at the ends is loaded as shown in the Figure. Find (i) moment at supports (ii) reactions at the supports. Draw the B.M. and S.F. diagrams. (May/June 2013)



A.U. Question paper problems

- A continuous beam ABCD 20 m long is fixed at A, simply supported at D and carried on the supports B and C at 5 m and 12 m from the left end A. It carries two concentrated loads of 80 kN and 40 kN at 3 m and 8 m respectively from A and uniformly distributed load of 12 kN/m over the span CD. Analyse the beam by theorem of three moments and draw the shear force and bending moment diagrams. (May/June 2012)

2 marks Questions and Answers

1. Explain briefly about fixed end moments.
2. Define theorem of three moments.
3. What is meant by prop ?
4. What is the value of prop reaction in a propped cantilever of span ‘L’, when it is subjected to a u.d.l over the entire length?
5. What are the advantages and limitations of theorem of three moments?
6. Determine the prop reaction for a cantilever beam with udl over entire span.
7. Write the three moment equation, stating all the variables used.