Chapter 3

Propped Cantilever and Fixed Beams

1 Introduction

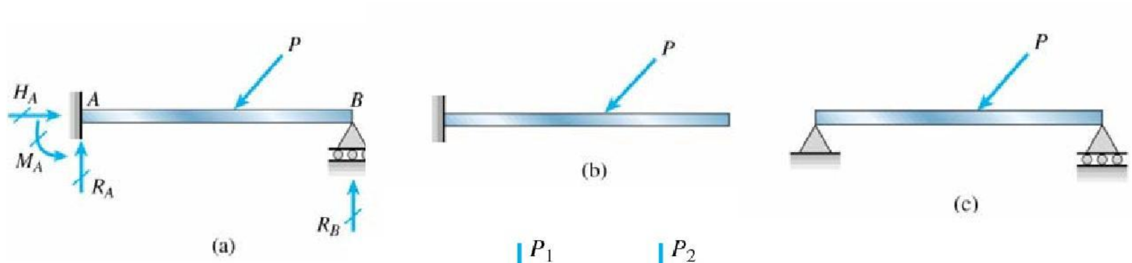
in this chapter we will analyze the beam in which the number of reactions exceed the number of independent equations of equilibrium

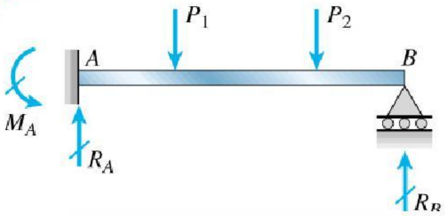
integration of the differential equation, method of superposition

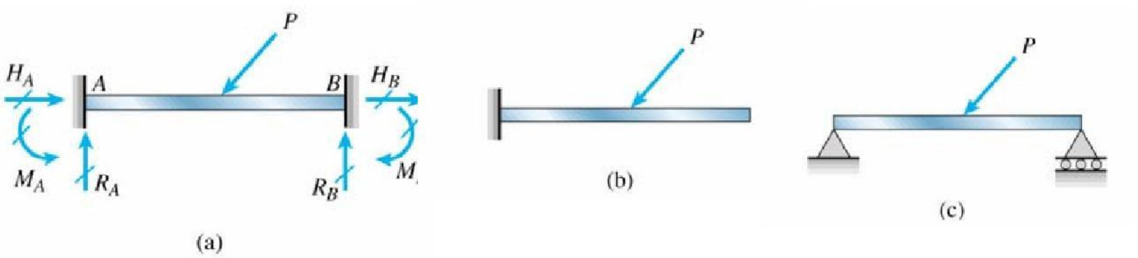
compatibility equation (consistence of deformation)

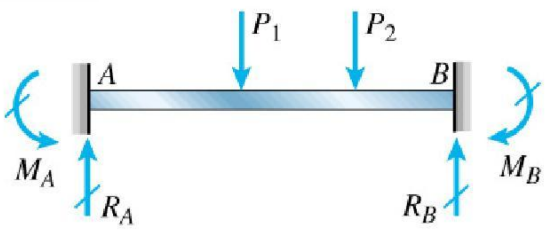
10.2 Types of Statically Indeterminate Beams

the number of reactions in excess of the number of equilibrium equations is called the degree of static indeterminacy









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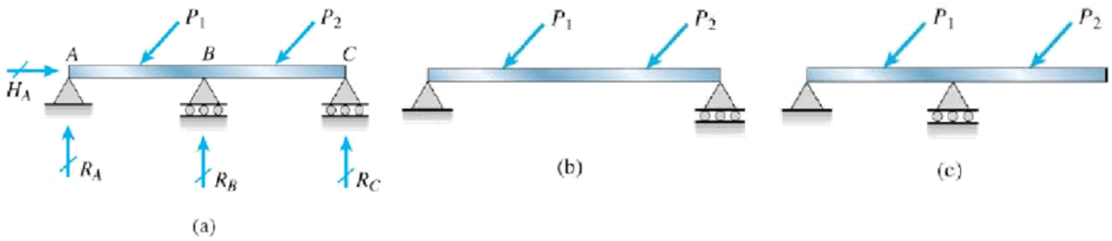


the excess reactions are called static redundants

the structure that remains when the redundants are released is called released structure or the primary structure

10.3 Analysis by the Differential Equations of the Deflection Curve

$$EIv'' = M \qquad EIv''' = V \qquad EIv^{iv} = -q$$

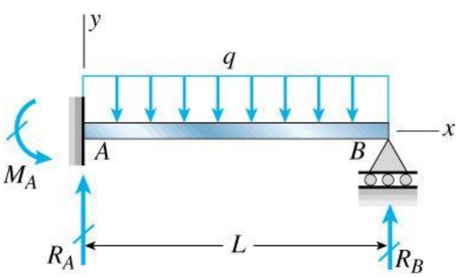
the procedure is essentially the same as that for a statically determine beam and consists of writing the differential equation, integrating to obtain its general solution, and then applying boundary and other conditions to evaluate the unknown quantities, the unknowns consist of the redundant reactions as well as the constants of integration

this method have the computational difficulties that arise when a large number of constants to be evaluated, it is practical only for relatively simple case

Example 10-1

a propped cantilever beam *AB* supports a uniform load *q*

determine the reactions, shear forces, bending moments, slopes, and deflections



choose R_B as the redundant, then

$$R_A = qL - R_B \qquad M_A = \frac{qL^2}{2} - R_B L$$

and the bending moment of the beam is

$$M = R_A x - M_A - \frac{qx^2}{2}$$

$$= qLx - R_B x - \frac{qL^2}{2} - R_B L - \frac{qx^2}{2}$$

$$EIv'' = M = qLx - R_B x - \frac{qL^2}{2} - R_B L - \frac{qx^2}{2}$$

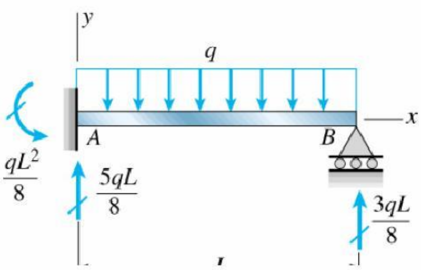
$$EIv' = \frac{qLx^2}{2} - \frac{R_B x^2}{2} - \frac{qL^2 x}{2} - R_B L x - \frac{qx^3}{6} + C_1$$

$$EIv = \frac{qLx^3}{6} - \frac{R_b x^3}{6} - \frac{qL^2 x^2}{4} - \frac{R_B L x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

boundary conditions

$$v(0) = 0 \qquad v'(0) = 0 \qquad v(L) = 0$$

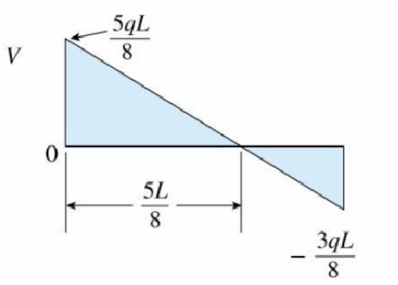
it is obtained



$$C_1 = C_2 = 0 \qquad R_B = 3qL/8$$

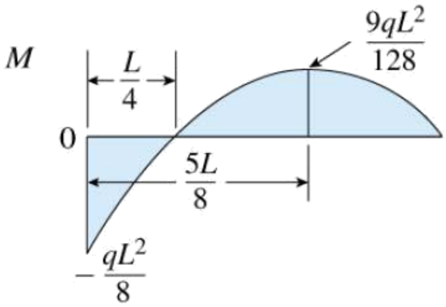
and $R_A = 5qL/8$

$$M_A = qL^2/8$$



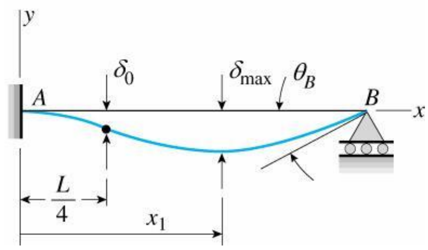
the shear force and bending moment are

$$V = R_A - qx = \frac{5qL}{8} - qx$$



$$M = R_A x - M_A - \frac{qx^2}{2}$$

$$= \frac{5qLx}{8} - \frac{qL^2}{8} - \frac{qx^2}{2}$$



the maximum shear force is

$V_{max} = 5qL/8$ at the fixed end

the maximum positive and negative moments are

$M_{pos} = 9qL^2/128$ $M_{neg} = -qL^2/8$

slope and deflection of the beam

$$v' = \frac{qx}{48EI}(-6L^2 + 15Lx - 8x^2)$$

$$v = -\frac{qx^2}{48EI}(3L^2 - 5Lx + 2x^2)$$

to determine the $_{max}$, set $v' = 0$

$$-6L^2 + 15Lx - 8x^2 = 0$$

we have $x_1 = 0.5785L$

$$_{max} = -v(x_1) = 0.005416 \frac{qL^4}{EI}$$

the point of inflection is located at $M = 0$, i.e. $x = L/4$

< 0 and $M < 0$ for $x < L/4$

> 0 and $M > 0$ for $x > L/4$

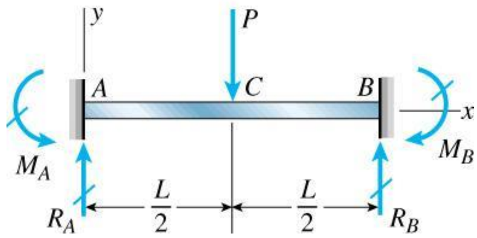
the slope at B is

$$_B = (y')_{x=L} = \frac{qL^3}{48EI}$$

Example 10-2

a fixed-end beam ABC supports a concentrated load P at the midpoint

determine the reactions, shear forces, bending moments, slopes, and deflections



because the load P in vertical direction and symmetric

$$H_A = H_B = 0 \qquad R_A = R_B = P/2$$

$$M_A = M_B \qquad \text{(1 degree of indeterminacy)}$$

$$M = \frac{Px}{2} - M_A \quad (0 \leqq x \leqq L/2)$$

$$EIv'' = M = \frac{Px}{2} - M_A \quad (0 \leqq x \leqq L/2)$$

after integration, it is obtained

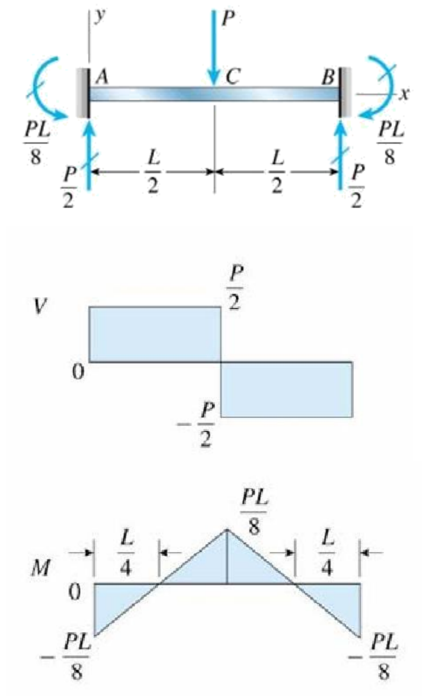
$$EIv' = \frac{Px^2}{4} - M_A x + C_1 \quad (0 \leqq x \leqq L/2)$$

$$EIv = \frac{Px^3}{12} - \frac{M_A x^2}{2} + C_1 x + C_2 \quad (0 \leqq x \leqq L/2)$$

boundary conditions

$$v(0) = 0 \qquad v'(0) = 0$$

symmetric condition



$v'(0) = 0$

the constants C_1, C_2 and the moment M_A are obtained

$$C_1 = C_2 = 0$$

$$M_A = \frac{PL}{8} = M_B$$

the shear force and bending moment diagrams can be plotted

thus the slope and deflection equations are

$$v' = -\frac{Px}{8EI}(L - 2x) \quad (0 \leqq x \leqq L/2)$$

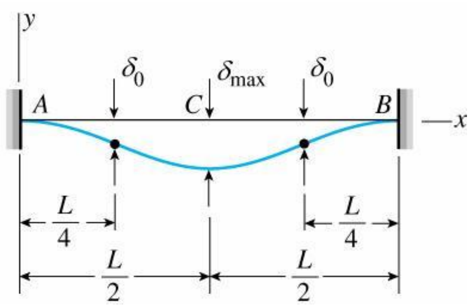
$$v = -\frac{Px^2}{48EI}(3L - 4x) \quad (0 \leqq x \leqq L/2)$$

the maximum deflection occurs at the center

$$_{max} = -v(L/2) = \frac{PL^3}{192EI}$$

the point of inflection occurs at the point where $M = 0$, i.e. $x = L/4$, the deflection at this point is

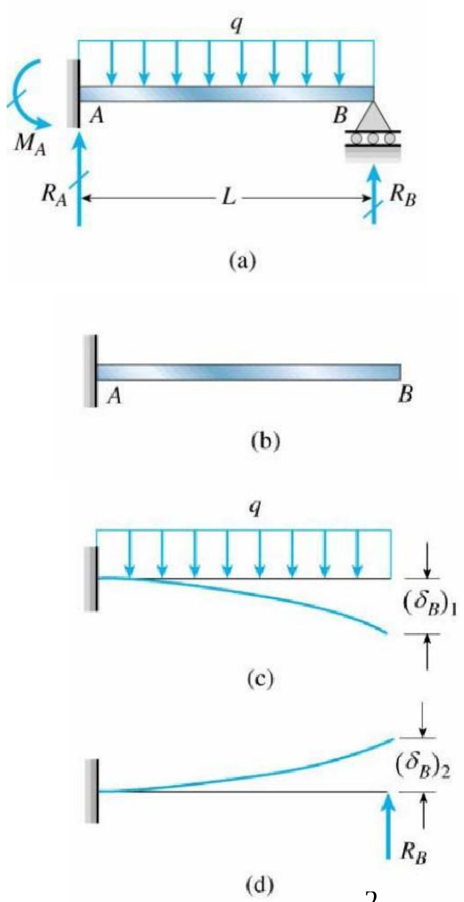
$$= -v(L/4) = \frac{PL^3}{384EI}$$



which is equal $_{max}/2$

10.4 Method of Superposition

1. selecting the reaction redundants
2. establish the force-displacement relations
3. consistence of deformation (compatibility equation)



consider a propped cantilever beam

(i) select R_B as the redundant, then

$$R_A = qL - R_B \qquad M_A = \frac{qL^2}{2} - R_B L$$

force-displacement relation

$$(_B)_1 = \frac{qL^4}{8EI} \qquad (_B)_2 = \frac{R_B L^3}{3EI}$$

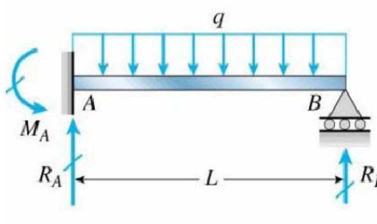
compatibility equation

$$_B = (_B)_1 - (_B)_1 = 0$$

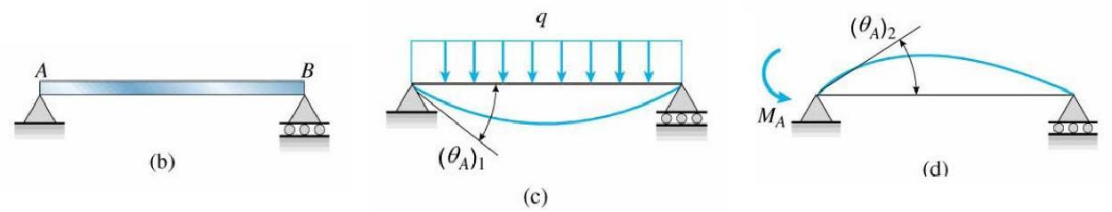
$$\frac{qL^4}{8EI} = \frac{R_B L^3}{3EI}$$

$$R_B = \frac{3qL}{8} \Rightarrow R_A = \frac{5qL}{8} \qquad M_A = \frac{qL^2}{8}$$

(ii) select the moment M_A as the redundant



$$R_A = \frac{qL}{2} + \frac{M_A}{L} \qquad R_B = \frac{qL}{2} - \frac{M_A}{L}$$



force-displacement relation

$$(\ _A)_1 = \frac{qL^3}{24EI} \qquad (\ _A)_2 = \frac{M_A L}{3EI}$$

compatibility equation

$$_A = (\ _A)_1 - (\ _A)_2 = \frac{qL^3}{24EI} - \frac{M_A L}{3EI} = 0$$

thus $M_A = qL^2/8$

and $R_A = 5qL/8 \qquad R_B = 3qL/8$

Example 10-3

a continuous beam *ABC* supports a uniform load *q*

determine the reactions

select R_B as the redundant, then

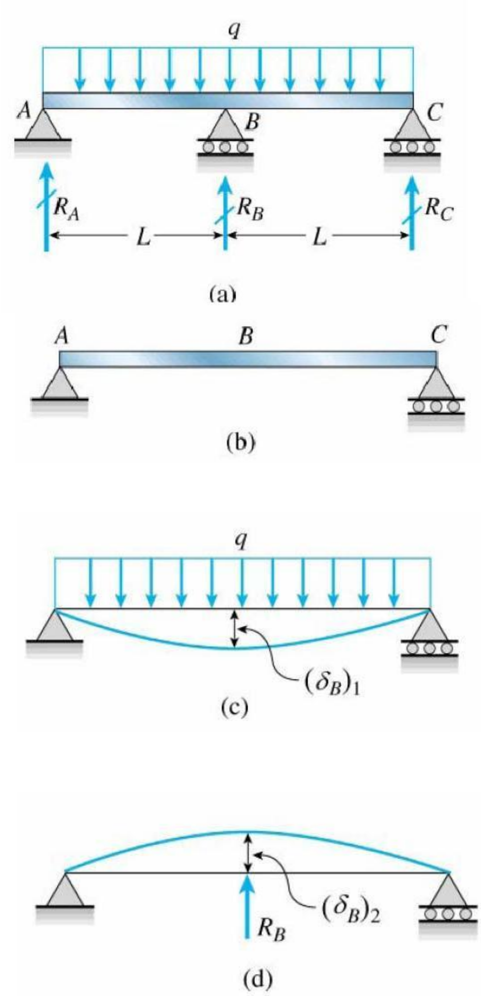
$$R_A = R_C = qL - \frac{qL}{2}$$

force-displacement relation

$$(\ _B)_1 = \frac{5qL(2L)^4}{384EI} = \frac{5qL^4}{24EI}$$

$$(\ _B)_2 = \frac{R_B(2L)^3}{48EI} = \frac{R_B L^3}{6EI}$$

compatibility equation



$$_B = (_B)_1 - (_B)_2 = \frac{5qL^4}{24EI} - \frac{R_BL^3}{6EI} = 0$$

thus $R_B = 5qL/4$

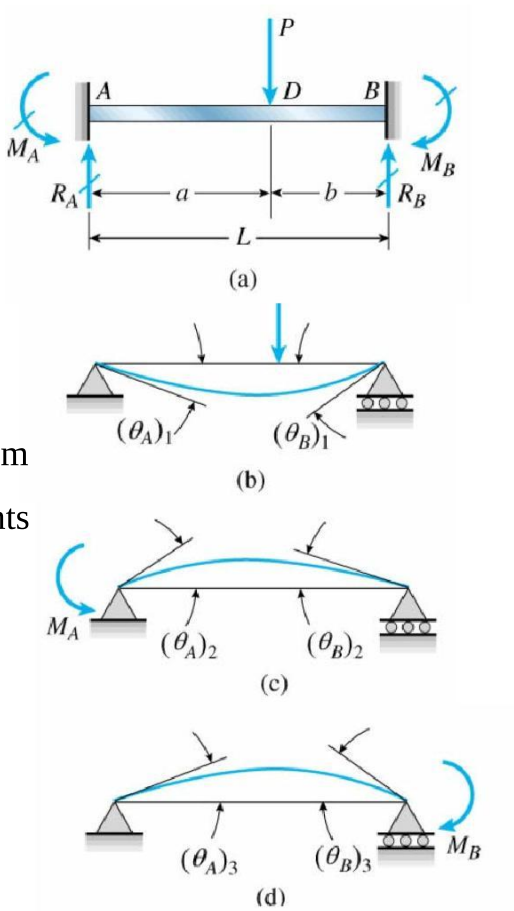
and $R_A = R_C = 3qL/8$

Example 10-4

a fixed-end beam AB is loaded by a force P acting at point D

determine reactions at the ends

also determine $_D$



this is a 2-degree of indeterminacy problem

select M_A and M_B as the redundants

$$R_A = \frac{Pb}{L} + \frac{M_A}{L} - \frac{M_B}{L}$$

$$R_B = \frac{Pa}{L} - \frac{M_A}{L} + \frac{M_B}{L}$$

force-displacement relations

$$(_A)_1 = \frac{Pab(L + b)}{6LEI} \qquad (_B)_1 = \frac{Pab(L + a)}{6LEI}$$

$$(_A)_2 = \frac{M_AL}{3EI} \quad (_B)_2 = \frac{M_AL}{6EI}$$

$$(_A)_3 = \frac{M_BL}{6EI} \quad (_B)_3 = \frac{M_BL}{3EI}$$

compatibility equations

$$A = (A)_1 - (A)_2 - (A)_3 = 0$$

$$B = (B)_1 - (B)_2 - (B)_3 = 0$$

i.e. $$\frac{M_A L}{3EI} + \frac{M_B L}{6EI} = \frac{Pab(L + b)}{6LEI}$$

$$\frac{M_A L}{6EI} + \frac{M_B L}{3EI} = \frac{Pab(L + a)}{6LEI}$$

solving these equations, we obtain

$$M_A = \frac{Pab^2}{L^2} \qquad M_B = \frac{Pa^2b}{L^2}$$

and the reactions are

$$R_A = \frac{Pb^2}{L^3}(L + 2a) \qquad R_B = \frac{Pa^2}{L^3}(L + 2b)$$

the deflection $_D$ can be expressed as

$$D = (D)_1 - (D)_2 - (D)_3$$

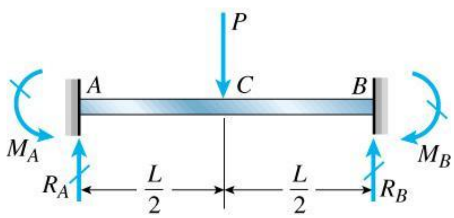
$$(D)_1 = \frac{Pa^2b^2}{3LEI}$$

$$(D)_2 = \frac{M_A ab}{6LEI}(L + b) = \frac{Pa^2b^3}{6L^3EI}(L + b)$$

$$(D)_3 = \frac{M_B ab}{6LEI}(L + a) = \frac{Pa^3b^2}{6L^3EI}(L + a)$$

thus $$D = \frac{Pa^3b^3}{3L^3EI}$$

if $a = b = L/2$

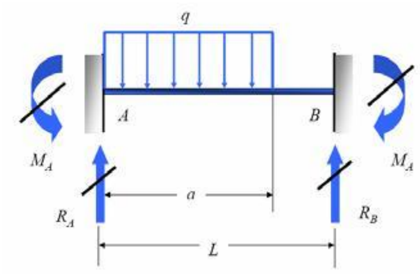


then $M_A = M_B = \dfrac{PL}{8}$ $\qquad R_A = R_B = \dfrac{P}{2}$

and $\delta_C = \dfrac{PL^3}{192EI}$

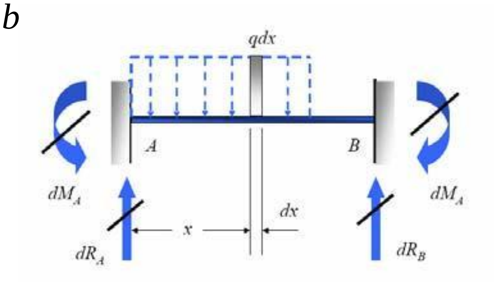
Example 10-5

a fixed-end beam *AB* supports a uniform load *q* acting over part of the span



determine the reactions of the beam

to obtain the moments caused by *qdx*, replace *P* to *qdx*, *a* to *x*, and *b* to *L - x*



$$dM_A = \frac{qx(L-x)^2 dx}{L^2}$$

$$dM_B = \frac{qx^2(L-x)dx}{L^2}$$

integrating over the loaded part

$$M_A = \int dM_A = \frac{q}{L^2}\int_0^a x(L-x)^2 dx = \frac{qa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = \int dM_B = \frac{q}{L^2}\int_0^a x^2(L-x)dx = \frac{qa^3}{12L^2}(4L^2 - 3a)$$

Similarly

$$dR_A = \frac{q(L-x)^2(L+2x)dx}{L^3}$$

$$dR_B = \frac{qx^2(3L-2x)dx}{L^3}$$

integrating over the loaded part

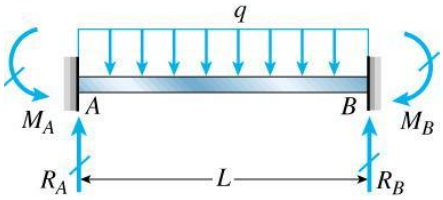
$$R_A = \int dR_A = \frac{q}{L^3}\int_0^a (L-x)^2(L+2x)dx = \frac{qa}{2L^3}(2L^3 - 2a^2L + a^3)$$

$$R_B = \int dR_B = \frac{q}{L^3}\int_0^a x^2(3L-2x)dx = \frac{qa^3}{2L^3}(2L-a)$$

for the uniform acting over the entire length, i.e. $a = L$

$$M_A = M_B = \frac{qL^2}{12}$$

$$R_A = R_B = \frac{qL}{2}$$



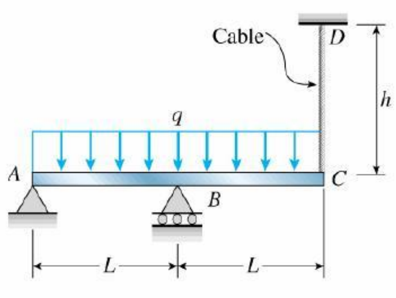
the center point deflections due to uniform load and the end moments are

$$(\delta_C)_1 = \frac{5qL^4}{384EI} \qquad (\delta_C)_2 = \frac{M_AL}{8EI} = \frac{(qL^2/12)L^2}{8EI} = \frac{qL^4}{96EI}$$

$$\delta_C = (\delta_C)_1 - (\delta_C)_2 = \frac{qL^4}{384EI}$$

Example 10-6

a beam ABC rests on supports A and B and is supported by a cable at C

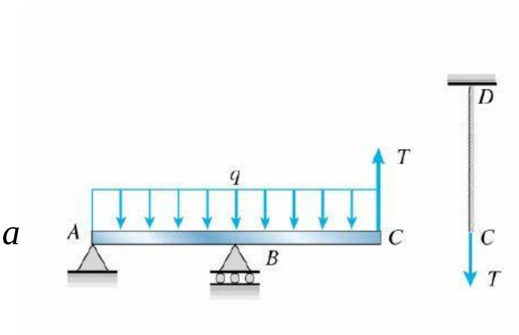


(a)

find the force T of the cable

take the cable force T as redundant

the deflection $(_C)_1$ due the uniform load can be found from example 9.9 with $a = L$



(b)

$$(_C)_1 = \frac{qL^4}{4E_bI_b}$$

the deflection $(_C)_2$ due to a force T acting on C is obtained

use conjugate beam method

$$(_C)_2 = M = \frac{TL^2}{3E_bI_b} L + \frac{TL}{E_bI_b} \frac{L}{2} \frac{2L}{3}$$

$$= \frac{2TL^3}{3E_bI_b}$$

the elongation of the cable is

$$(_C)_3 = \frac{Th}{E_cA_c}$$

compatibility equation

$$(_C)_1 - (_C)_2 = (_C)_3$$

$$\frac{qL^4}{4E_bI_b} - \frac{2TL^3}{3E_bI_b} = \frac{Th}{E_cA_c}$$

$$T = \frac{3qL^4E_cA_c}{8L^3E_cA_c + 12hE_bI}$$