Stress and Strain

(a) Inadequate Strength

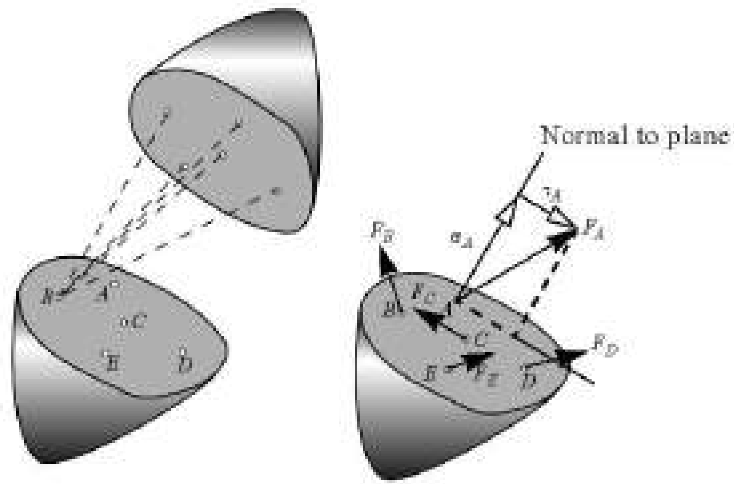
(b) Adequate Stiffness

Courtesy (a) *Rijon*. http://en.wikipedia.org/wiki/File:Dhaka_Savar_Building_Collapse.jpg (b) *Mate 2nd Class Jason Sellers III*. http://commons.wikimedia.org/wiki/File:Diving.jpg

The learning objectives in this chapter are:

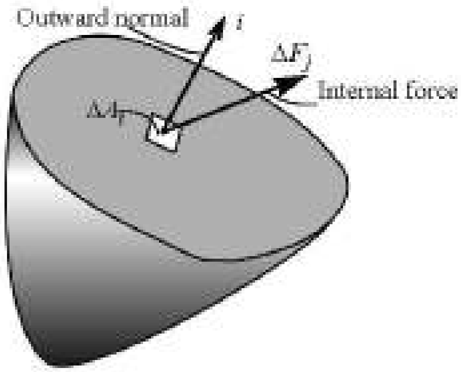
- To understand the concepts of stress and strain.
- To understand stress and strain transformations in three dimensions.
- To understand the relationship of stress to internal forces and moments.

Internally Distributed Force System



- The intensity of internal distributed forces on an imaginary cut surface of a body is called the *stress on a surface*.
- The intensity of internal distributed force that is normal to the surface of an imaginary cut is called the *normal stress* on a surface.
- The intensity of internal distributed force that is parallel to the surface of an imaginary cut surface is called the *shear stress* on the surface.

Stress at a Point



$$\sigma_{ij} = \lim_{\Delta A_i \to 0} \left(\frac{\Delta F_j}{\Delta A_i} \right)$$

direction of outward normal to the imaginary cut surface.

direction of the internal force component.

- ΔA_i will be considered positive if the outward normal to the surface is in the positive i direction.
- A stress component is positive if numerator and denominator have the same sign. Thus σ_{ij} is positive if: (1) ΔF_j and ΔA_i are both positive. (2) ΔF_j and ΔA_i are both negative.

- Stress Matrix in 3-D: $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$

Table 1.1. Comparison of number of components

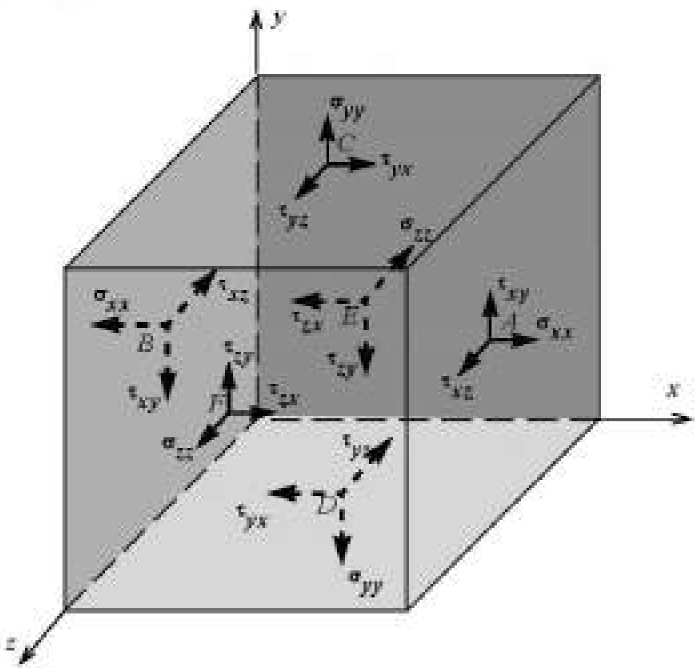
Quantity	1-D	2-D	3-D
Scaler	$1=1^0$	$1=2^0$	$1=3^0$
Vector	$1=1^1$	$2=2^1$	$3=3^1$
Stress	$1=1^2$	$4=2^2$	$9=3^2$

Stress Element

- Stress element is an imaginary object that helps us visualize stress at a point by constructing surfaces that have outward normal in the coordinate directions.

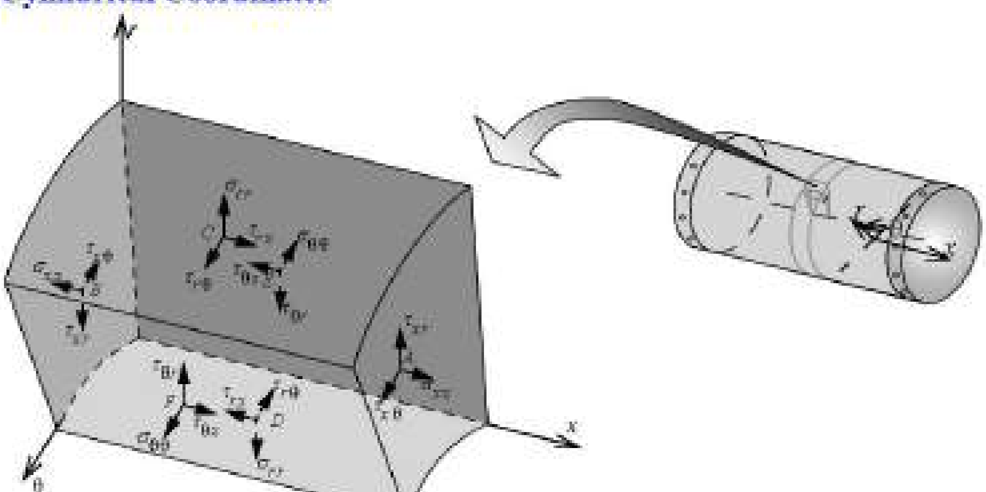
Stress Element in Cartesian Coordinates

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



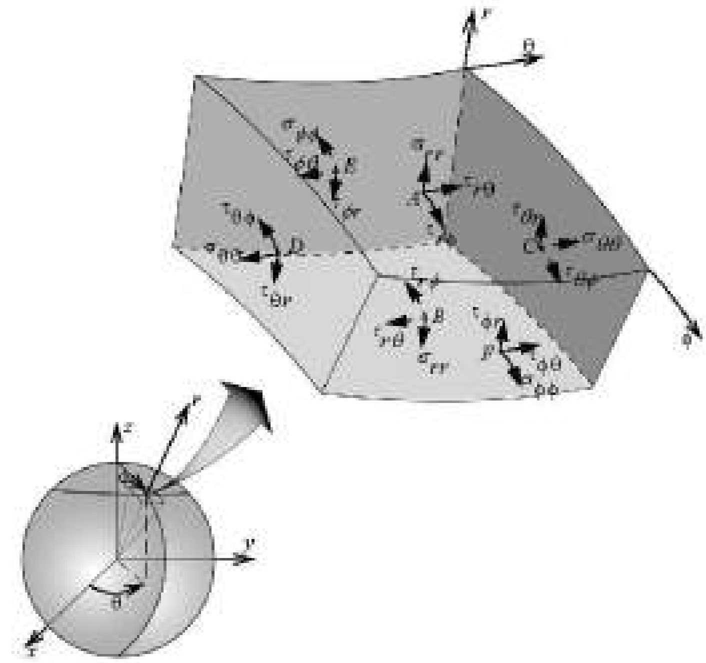
Stress Element in Cylindrical Coordinates

$$\begin{bmatrix} \sigma_{xx} & \tau_{xr} & \tau_{x\theta} \\ \tau_{rx} & \sigma_{rr} & \tau_{r\theta} \\ \tau_{\theta x} & \tau_{\theta r} & \sigma_{\theta\theta} \end{bmatrix}$$



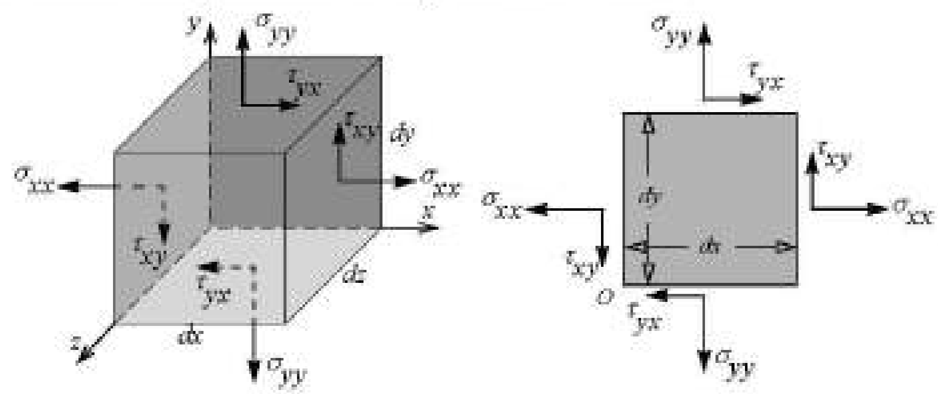
Stress Element in Spherical Coordinates

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$



Plane Stress: All stress components on a plane are zero.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Symmetric Shear Stresses: $\tau_{xy} = \tau_{yx}$ $\tau_{yz} = \tau_{zy}$ $\tau_{zx} = \tau_{xz}$

- A pair of symmetric shear stress points towards the corner or away from the corner.

C1.1 Show the non-zero stress components on the A,B, and C faces of the cube shown below.

$$\begin{bmatrix} \sigma_{xx} = 0 & \tau_{xy} = -15ksi & \tau_{xz} = 0 \\ \tau_{yx} = -15ksi & \sigma_{yy} = 10ksi(C) & \tau_{yz} = 25ksi \\ \tau_{zx} = 0 & \tau_{zy} = 25ksi & \sigma_{zz} = 20ksi(T) \end{bmatrix}$$

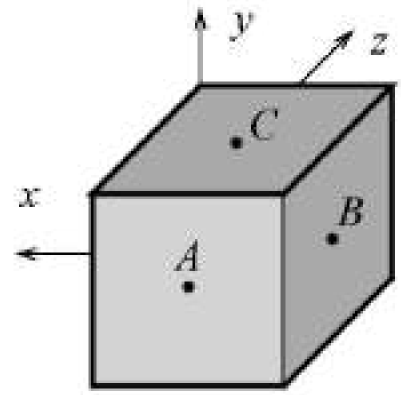
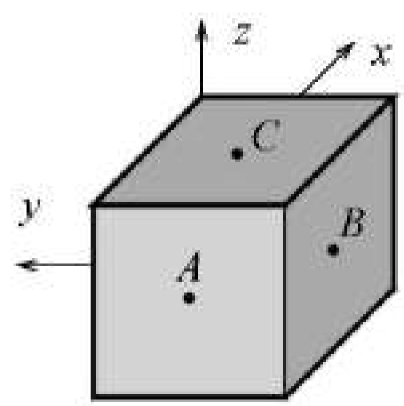


Fig. P1.1

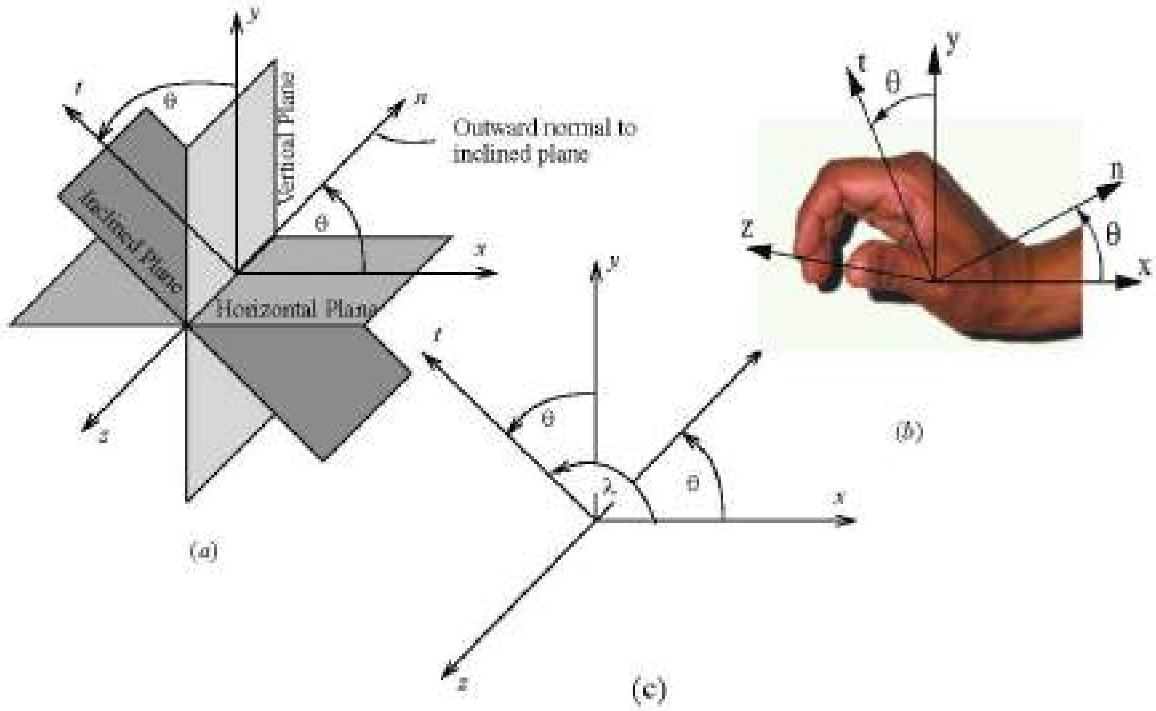
Class Problem 1.1

Show the non-zero stress components on the A,B, and C faces of the cube shown below.

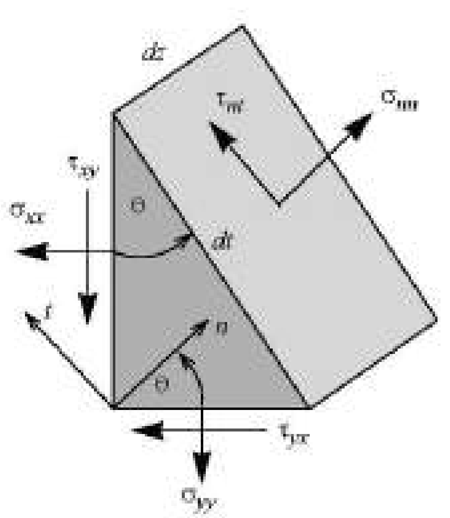
$$\begin{bmatrix} \sigma_{xx} = 0 & \tau_{xy} = -15ksi & \tau_{xz} = 0 \\ \tau_{yx} = -15ksi & \sigma_{yy} = 10ksi(C) & \tau_{yz} = 25ksi \\ \tau_{zx} = 0 & \tau_{zy} = 25ksi & \sigma_{zz} = 20ksi(T) \end{bmatrix}$$



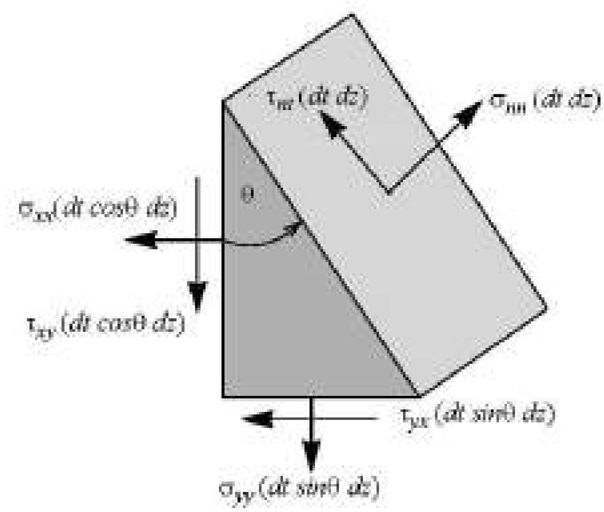
Stress transformation in two dimension



Stress Wedge



Force Wedge



$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{tt} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

Matrix Notation

$$n_x = \cos\theta \qquad n_y = \sin\theta \qquad t_x = \cos\lambda \qquad t_y = \sin\lambda$$

True only in 2D: $\lambda = 90 + \theta$; $t_x = -n_y \qquad t_y = n_x$

$$\{n\} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \qquad \{t\} = \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

The symmetry of shear stresses $\qquad [\sigma]^T = [\sigma]$

$$\boxed{\sigma_{nn} = \{n\}^T[\sigma]\{n\}}$$

$$\boxed{\tau_{nt} = \{t\}^T[\sigma]\{n\}}$$

$$\boxed{\sigma_{tt} = \{t\}^T[\sigma]\{t\}}$$

Traction or Stress vector

Mathematically the stress vector {S} is defined as:

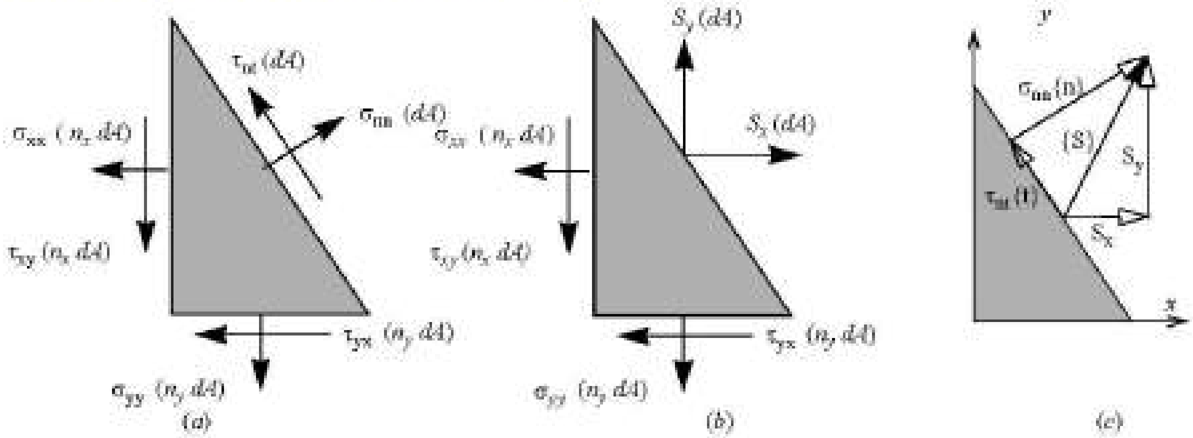
$$\{S\} = [\sigma]\{n\}$$

$$S_x = \sigma_{xx}n_x + \tau_{xy}n_y$$

$$S_y = \tau_{yx}n_x + \sigma_{yy}n_y$$

- pressure is a scaler quantity.
- traction is a vector quantity.
- stress is a second order tensor.

Statically equivalent force wedge.



Stress vector in different coordinate systems. $\{S\} = \sigma_{nn}\{n\} + \tau_{nt}\{t\}$

Principal Stresses and Directions

$$\{S\} = [\sigma]\{p\} = \sigma_p\{p\}$$

OR

$$\{S\} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{bmatrix} \sigma_p & 0 \\ 0 & \sigma_p \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

OR

$$\begin{bmatrix} (\sigma_{xx} - \sigma_p) & \tau_{xy} \\ \tau_{yx} & (\sigma_{yy} - \sigma_p) \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = 0$$

Characteristic equation

$$\sigma_p^2 - \sigma_p(\sigma_{xx} + \sigma_{yy}) + (\sigma_{xx}\sigma_{yy} - \tau_{xy}^2) = 0$$

Roots: $\sigma_{1,2} = [(\sigma_{xx} + \sigma_{yy}) \pm \sqrt{(\sigma_{xx} + \sigma_{yy})^2 - 4(\sigma_{xx}\sigma_{yy} - \tau_{xy}^2)}]/2$

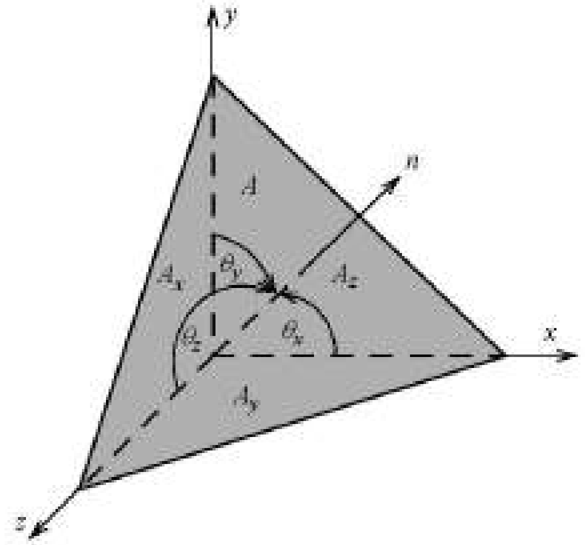
OR

$$\sigma_{1,2} = \left[\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}\right]$$

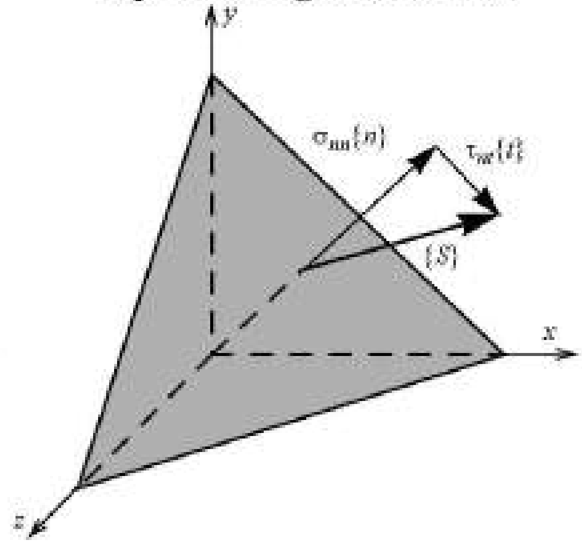
- The *eigenvalues* of the stress matrix are the principal stresses.
- The *eigenvectors* of the stress matrix are the principal directions.

Stress Transformation in 3-D

Direction cosines of a unit normal



Equilibrating shear stress



$$\{n\} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \qquad \{S\} = \begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\boxed{\sigma_{nn} = \{n\}^T[\sigma]\{n\}}$$

$$\boxed{\tau_{nt} = \{t\}^T[\sigma]\{n\}}$$

$$\boxed{\sigma_{tt} = \{t\}^T[\sigma]\{t\}}$$

$$\{S\} = [\sigma]\{n\}$$

Equilibrium condition: $\{S\} = \sigma_{nn}\{n\} + \tau_{nt}\{t_E\}$ implies $|S|^2 = \sigma_{nn}^2 + \tau_{nt}^2$

Principal Stresses and Directions

- Planes on which the shear stresses are zero are called the principal planes.
- The normal direction to the principal planes is referred to as the principal direction or the principal axis.
- The angles the principal axis makes with the global coordinate system are called the principal angles.

$$\{S\} = [\sigma]\{p\} = \sigma_p\{p\}$$

OR

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = \begin{bmatrix} \sigma_p & 0 & 0 \\ 0 & \sigma_p & 0 \\ 0 & 0 & \sigma_p \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

OR

$$\begin{bmatrix} (\sigma_{xx} - \sigma_p) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma_p) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma_p) \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = 0$$

- The *eigenvalues* of the stress matrix are the principal stresses.
- The *eigenvectors* of the stress matrix are the principal directions.

$$\boxed{p_x^2 + p_y^2 + p_z^2 = 1}$$

Principal stress convention

Ordered principal stresses in 3-D: $\sigma_1 > \sigma_2 > \sigma_3$

Ordered principal stresses in 2-D: $\sigma_1 > \sigma_2$

Principal Angles $0^o \le \theta_x, \theta_y, \theta_z \le 180^o$

Characteristic equation

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

Stress Invariants

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{zx} & \sigma_{zz} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{vmatrix}$$

$$x^3 - I_1 x^2 + I_2 x - I_3 = 0$$

Roots:

$$x_1 = 2A\cos\alpha + I_1/3 \qquad x_{2,3} = -2A\cos(\alpha \pm 60^o) + I_1/3$$

$$A = \sqrt{(I_1/3)^2 - I_2/3}$$

$$\cos 3\alpha = [2(I_1/3)^3 - (I_1/3)I_2 + I_3]/(2A^3)$$

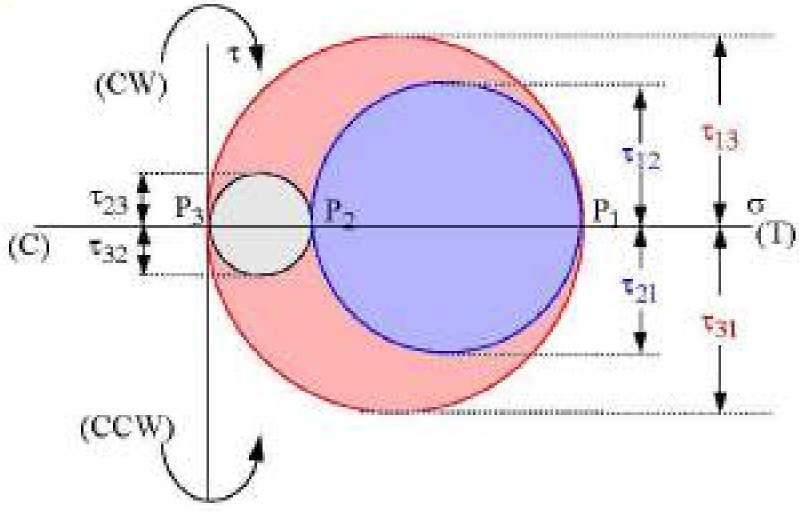
Principal Stress Matrix $[\sigma] = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

Maximum Shear Stress

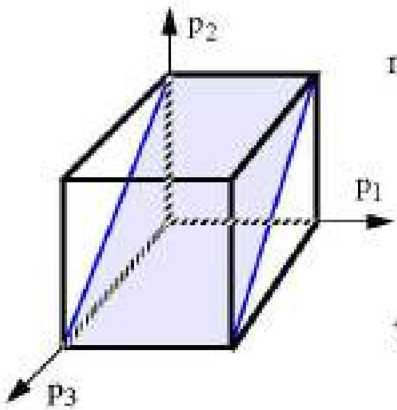
Plane Stress

$\sigma_3 = 0$



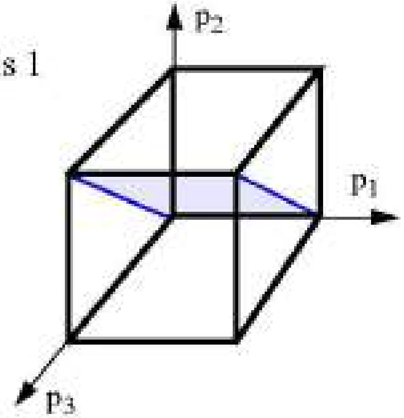
- maximum shear stress exists on two planes, each of which are 45° away from the principal planes.

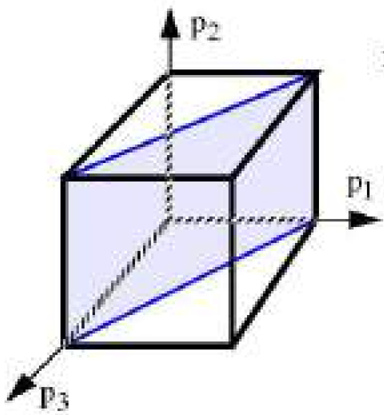
$$\tau_{max} = max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right)$$



rotation about principal axis 1

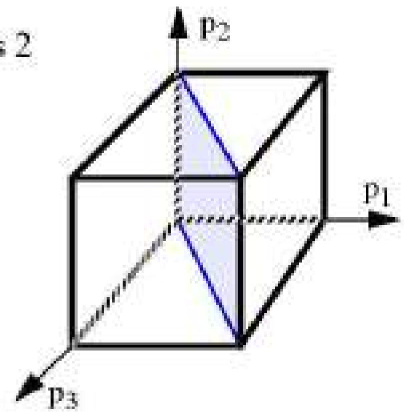
$$\tau_{23} = -\tau_{32} = \frac{\sigma_2 - \sigma_3}{2}$$

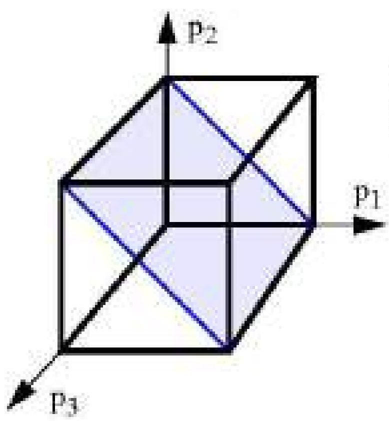




rotation about principal axis 2

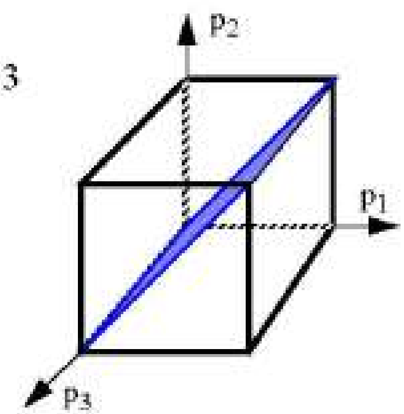
$$\tau_{31} = -\tau_{13} = \frac{\sigma_3 - \sigma_1}{2}$$





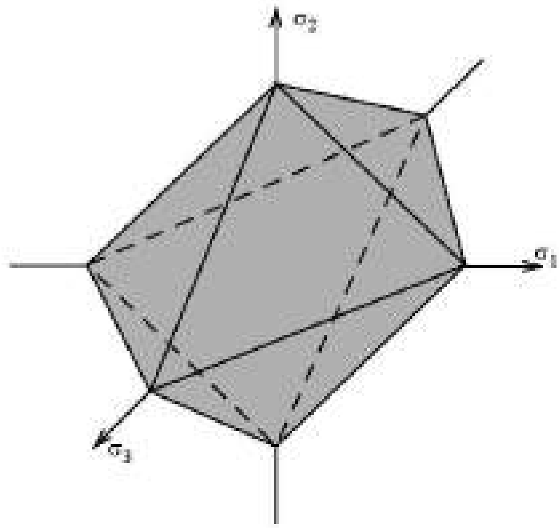
rotation about principal axis 3
(In-plane)

$$\tau_{21} = -\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}$$



Octahedral stresses

- A plane that makes equal angles with the principal planes is called an octahedral plane.
- The stresses on the octahedral planes are the octahedral stresses.



$$\{S\} = \begin{Bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{Bmatrix}$$

$$\sigma_{nn} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

$$\tau_{nt} = \sqrt{|S|^2 - \sigma_{nn}^2} = \sqrt{(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2) - \sigma_{nn}^2}$$

$$|n_1| = |n_2| = |n_3| = 1/\sqrt{3}$$

$$\sigma_{oct} = (\sigma_1 + \sigma_2 + \sigma_3)/3 = I_1/3$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Stress Deviators

Experiments have shown that hydrostatic pressure has negligible effect on the yield point until extreme high pressures are reached[1] (> 360 ksi). The high hydrostatic pressure does not effect the stress-strain curve in the elastic region but increase the ductility of the material, i.e., permits large plastic deformation before fracture.

Stress deviatoric matrix is the stress matrix from which the hydrostatic state of stress has been removed. The hydrostatic pressure (p) is given by

$$p = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{I_1}{3}$$

where, I_1 is the first stress invariant.

The stress deviatoric matrix in Cartesian coordinate principal coordinates is given by

Stress deviatoric matrices

$$\begin{bmatrix} \sigma_{xx} - \frac{I_1}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \frac{I_1}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \frac{I_1}{3} \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 - I_1/3 & 0 & 0 \\ 0 & \sigma_2 - I_1/3 & 0 \\ 0 & 0 & \sigma_3 - I_1/3 \end{bmatrix}$$

The deviatoric stress invariants are as given below

$$J_1 = 0$$

$$J_2 = \begin{vmatrix} \sigma_1 - I_1/3 & 0 \\ 0 & \sigma_2 - I_1/3 \end{vmatrix} + \begin{vmatrix} \sigma_2 - I_1/3 & 0 \\ 0 & \sigma_3 - I_1/3 \end{vmatrix} + \begin{vmatrix} \sigma_1 - I_1/3 & 0 \\ 0 & \sigma_3 - I_1/3 \end{vmatrix}$$

$$J_2 = I_2 - \frac{1}{3}I_1^2 = -\left(\frac{1}{6}\right)[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = -\left(\frac{3}{2}\right)\tau_{oct}^2$$

$$J_3 = I_3 - \frac{1}{3}I_1 I_2 + \frac{2}{27}I_1^3 = \frac{1}{27}(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2)$$

1. Mendelson A., "Plasticity: Theory and Applications", Macmillan Co., New York, (1968) section 2-5.

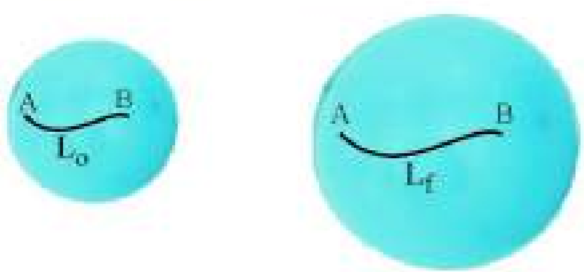
C1.2 The stress at a point is given by the stress matrix shown. Determine: (a) the normal and shear stress on a plane that has an outward normal at 37^o, 120^o, and 70.43^o, to x, y, and z direction respectively. (b) the principal stresses (c) the second principal direction and (d) the magnitude of the octahedral shear stress. (e) maximum shear stress (f) the deviatoric stress invariants.

$$\begin{bmatrix} 18 & 12 & 9 \\ 12 & 12 & -6 \\ 9 & -6 & 6 \end{bmatrix} ksi$$

Strain

- The total movement of a point with respect to a fixed reference coordinates is called *displacement*.
- The relative movement of a point with respect to another point on the body is called *deformation*.
- *Lagrangian strain* is computed from deformation by using the original undeformed geometry as the reference geometry.
- *Eulerian strain* is computed from deformation by using the final deformed geometry as the reference geometry.
- Relating strains to displacements is a problem in geometry.

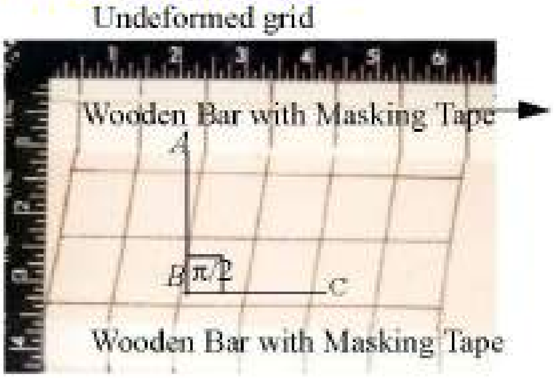
Average normal strain

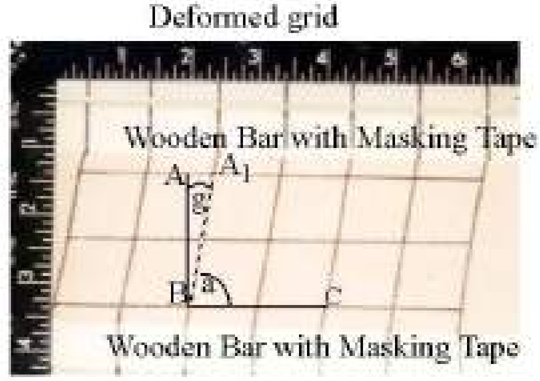


$$\varepsilon_{av} = \frac{L_f - L_o}{L_o} = \frac{\delta}{L_o}$$

- Elongations ($L_f > L_o$) result in *positive* normal strains. Contractions ($L_f < L_o$) result in *negative* normal strains.

Average shear strain





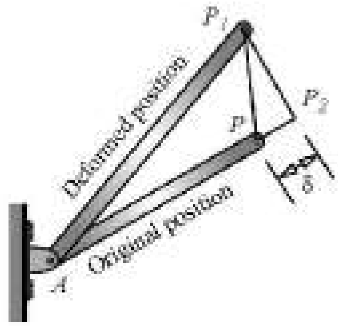
$$\gamma_{av} = \frac{\pi}{2} - \alpha$$

- Decreases in the angle ($\alpha < \pi / 2$) result in positive shear strain. Increase in the angle ($\alpha > \pi / 2$) result in negative shear strain

Units of average strain

- To differentiate average strain from strain at a point.
- in/in, or cm/cm, or m/m (for normal strains)
- rads (for shear strains)
- percentage. 0.5% is equal to a strain of 0.005
- prefix: $\mu = 10^{-6}$. 1000 μ in / in is equal to a strain 0.001 in /

Small Strain Approximation



$$L_f = \sqrt{L_o^2 + D^2 + 2L_o D\cos\theta}$$

$$L_f = L_o\sqrt{1 + \left(\frac{D}{L_o}\right)^2 + 2\left(\frac{D}{L_o}\right)\cos\theta}$$

$$\varepsilon = \frac{L_f - L_o}{L_o} = \sqrt{1 + \left(\frac{D}{L_o}\right)^2 + 2\left(\frac{D}{L_o}\right)\cos\theta} - 1$$

$$\varepsilon_{small} = \frac{D\cos\theta}{L_o}$$

ε_{small}	ε	% error
1.0	1.23607	19.1
0.5	0.58114	14.0
0.1	0.10454	4.3
0.05	0.005119	2.32
0.01	0.01005	0.49
0.005	0.00501	0.25

- Small-strain approximation may be used for strains less than 0.01
- Small normal strains are calculated by using the deformation component in the original direction of the line element regardless of the orientation of the deformed line element.
- In small shear strain (γ) calculations the following approximation may be used for the trigonometric functions: $\tan\gamma \approx \gamma$ $\sin\gamma \approx \gamma$ $\cos\gamma \approx 1$
- Small-strain calculations result in linear deformation analysis.
- Drawing approximate deformed shape is very important in analysis of small strains.

C1.3 A roller at P slides in a slot as shown. Determine the deformation in bar AP and bar BP by using small strain approximation.

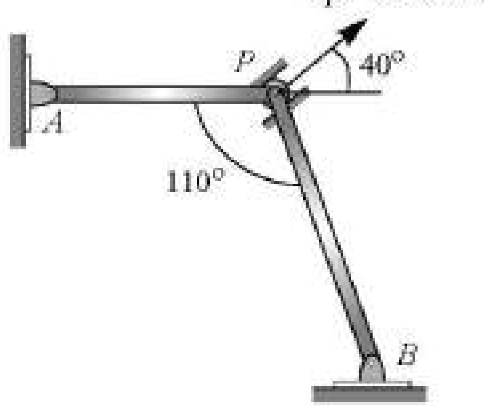
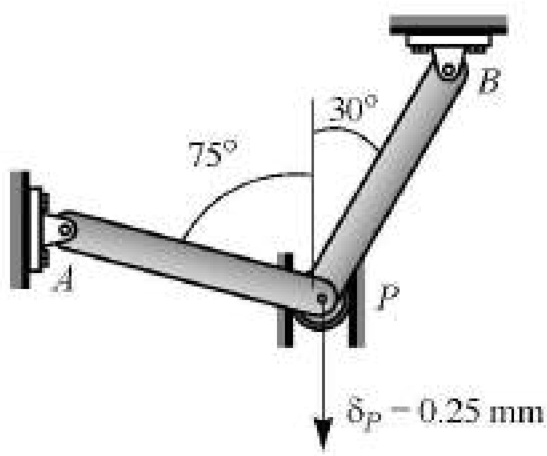


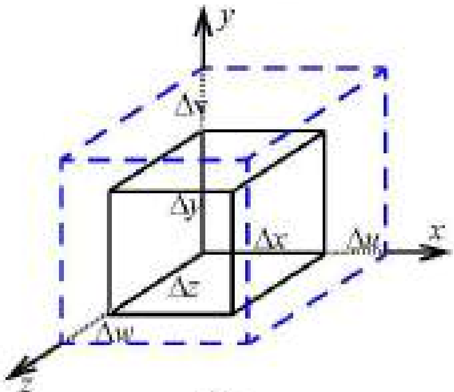
Fig. C1.3

Class Problem 1.2

Draw an approximate exaggerated deformed shape.
Using small strain approximation write equations relating δ_{AP} and δ_{BP} to δ_P.



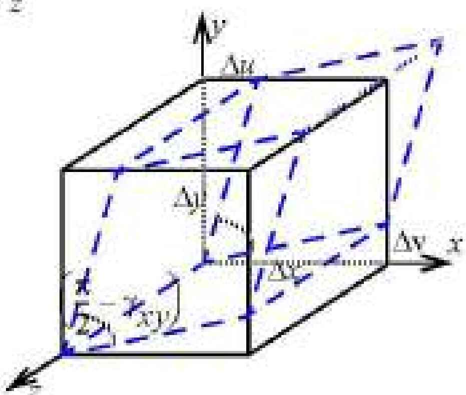
Engineering strain at a point



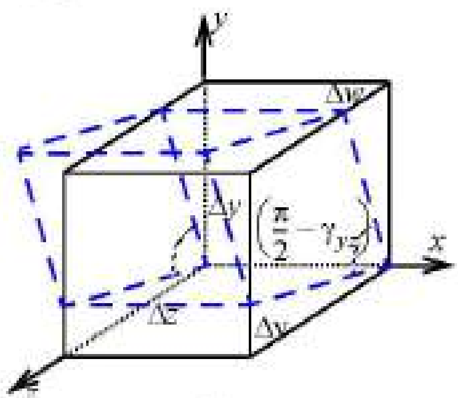
$$\varepsilon_{xx} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0}\left(\frac{\Delta v}{\Delta y}\right) = \frac{\partial v}{\partial y}$$

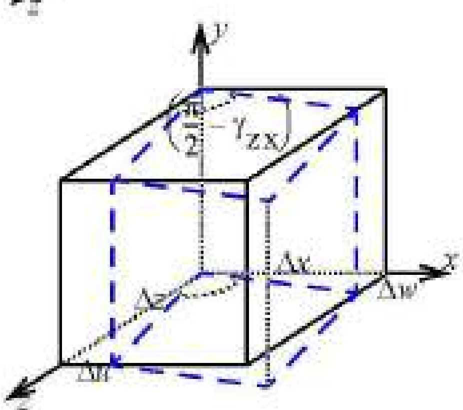
$$\varepsilon_{zz} = \lim_{\Delta z \to 0}\left(\frac{\Delta w}{\Delta z}\right) = \frac{\partial w}{\partial z}$$



$$\gamma_{xy} = \gamma_{yx} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



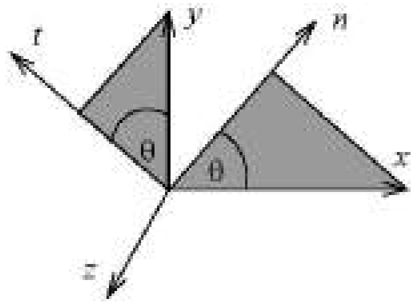
$$\gamma_{yz} = \gamma_{zy} = \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}}\left(\frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}\right) = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$



$$\gamma_{zx} = \gamma_{xz} = \lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}}\left(\frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}\right) = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Strain Transformation

Strain transformation is relating strains in two coordinate systems.



$$x = n\cos\theta - t\sin\theta \qquad y = n\sin\theta + t\cos\theta$$

$$u_n = u\cos\theta + v\sin\theta \qquad v_t = -u\sin\theta + v\cos\theta$$

$$\varepsilon_{nn} = \frac{\partial u_n}{\partial n} = \frac{\partial u_n}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial u_n}{\partial y}\frac{\partial y}{\partial n}$$

Strain transformation equations in 2-D

$$\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_{tt} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta$$

$$\gamma_{nt} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

Stress transformation equations in 2-D

$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_{tt} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

$$\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

- tensor normal strains = engineering normal strains
- tensor shear strains = (engineering shear strains)/ 2

Tensor strain matrix from engineering strains

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} = \gamma_{xy}/2 & \varepsilon_{xz} = \gamma_{xz}/2 \\ \varepsilon_{yx} = \gamma_{yx}/2 & \varepsilon_{yy} & \varepsilon_{yz} = \gamma_{yz}/2 \\ \varepsilon_{zx} = \gamma_{zx}/2 & \varepsilon_{zy} = \gamma_{zy}/2 & \varepsilon_{zz} \end{bmatrix}$$

$$\varepsilon_{nn} = \{n\}^T[\varepsilon]\{n\}$$

$$\varepsilon_{nt} = \{t\}^T[\varepsilon]\{n\} \qquad \gamma_{nt} = 2\varepsilon_{nt}$$

$$\varepsilon_{tt} = \{t\}^T[\varepsilon]\{t\}$$

Characteristic equation

$$\varepsilon_p^3 - I_1\varepsilon_p^2 + I_2\varepsilon_p - I_3 = 0$$

Strain invariants

$$I_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

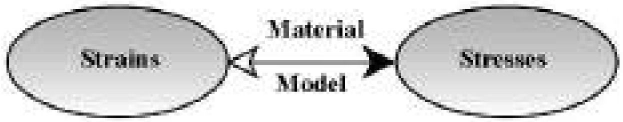
$$I_2 = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

$$I_3 = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_1\varepsilon_2\varepsilon_3$$

Maximum shear strain

$$\frac{\gamma_{max}}{2} = max\left(\left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right|, \left|\frac{\varepsilon_2 - \varepsilon_3}{2}\right|, \left|\frac{\varepsilon_3 - \varepsilon_1}{2}\right|\right)$$

Material Description



Linear Material Models

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

$$C_{ij} = C_{ji}$$

- The most general linear anisotropic material requires 21 independent constants.

Monoclinic material

- Has 1 plane of symmetry.
- If xy is the plane of symmetry then stress-strain relations in +ve & -ve z direction are the same.
- Requires 13 independent material constants.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

- x, y, z are the *material coordinate system*.
- The zero's in the C matrix can become non-zero in coordinate systems other than *material coordinate system*.

Orthotropic material

- Has two axis of symmetry.
- Requires 9 independent constants in 3_D.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

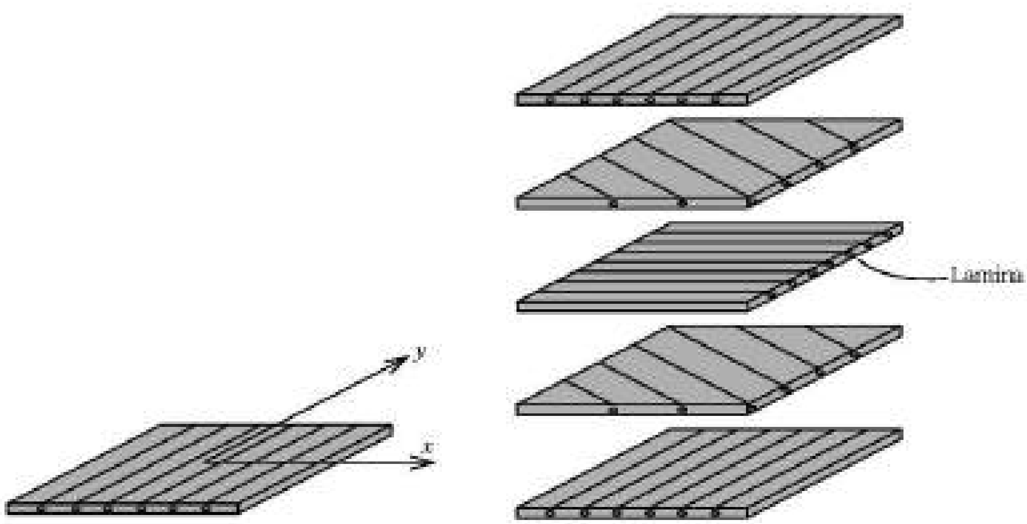
- *x, y, z* are the *material coordinate system.*
- The zero's in the C matrix can become non-zero in coordinate systems other than *material coordinate system.*

For plane stress problems (requires 4 independent constants)

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}$$

Long Fiber Composite

- Each lamina is an orthotropic material.
- A symmetric stacking about mid surface creates an orthotropic composite plate.



Transversely isotropic material

- Material is isotropic in a plane.
- Requires 5 independent material constants.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & \\ 0 & 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

- The zero's in the C matrix can become non-zero in coordinate systems other than *material coordinate system.*

Short Fiber Composite

Chopped fiber is sprayed on to a epoxy produces a transversely isotropic material. It is isotropic in the plane.

Isotropic Material

- An isotropic material has a stress-strain relationships that are independent of the orientation of the coordinate system at a point.
- An isotropic body requires only two independent material constants

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) & \\ 0 & 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Engineering Constants: $C_{11} = 1/E$, $C_{12} = -\nu/E$, and $2(C_{11}-C_{12}) = 1/G$

- E = Modulus of Elasticity
- G = Shear Modulus of Elasticity

- ν = Poisson's Ratio

Generalized Hooke's Law

$$\varepsilon_{xx} = [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E \qquad \gamma_{xy} = \tau_{xy}/G$$
$$\varepsilon_{yy} = [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E \qquad \gamma_{yz} = \tau_{yz}/G \qquad G = \frac{E}{2(1+\nu)}$$
$$\varepsilon_{zz} = [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E \qquad \gamma_{zx} = \tau_{zx}/G$$

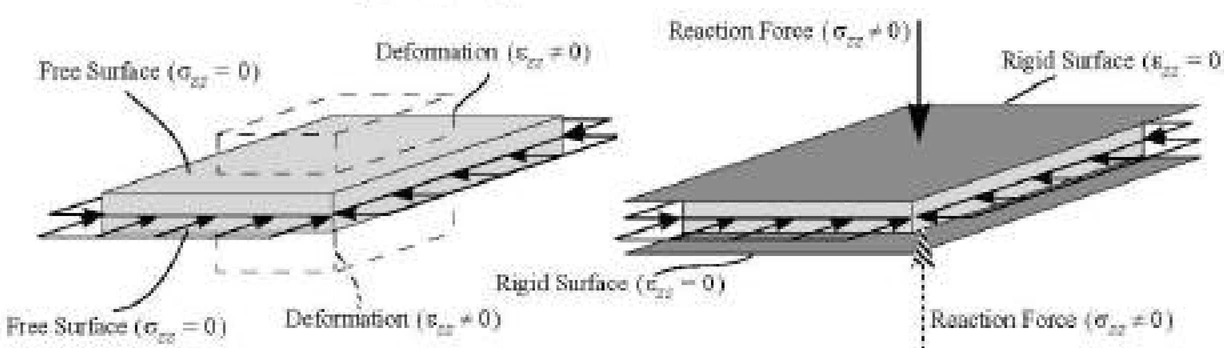
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$$

- Generalized Hooke's Law is valid for any orthogonal coordinate system.
- Principal direction for stress and strain are same *ONLY* for isotropic materials.
- A material is said to be homogeneous if the material properties are the same at all points on the body. Alternatively, if the material constants C_{ij} are functions of the coordinates x, y, or z, then the material is called non-homogeneous.

Plane Stress and Plane Strain.

$$\text{Plane Stress} \longrightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's Law}} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

$$\text{Plane Strain} \longrightarrow \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's Law}} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

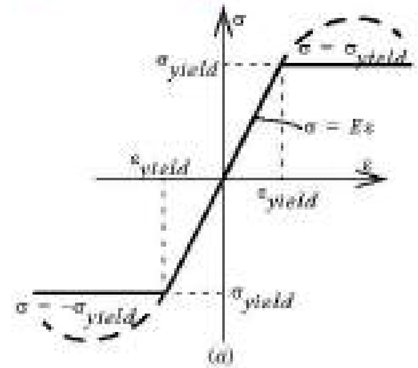


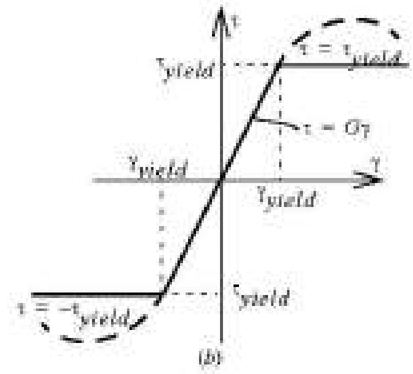
Non-linear material models

- Elastic-perfectly plastic in which the non-linearity is approximated by a constant.
- Linear strain hardening model (Bi-linear model) in which the non-linearity is approximated by a linear function.
- Power law model in which the non-linearity is approximated by one term non-linear function.

We will assume material behavior is same in tension and compression.

Elastic-perfectly plastic



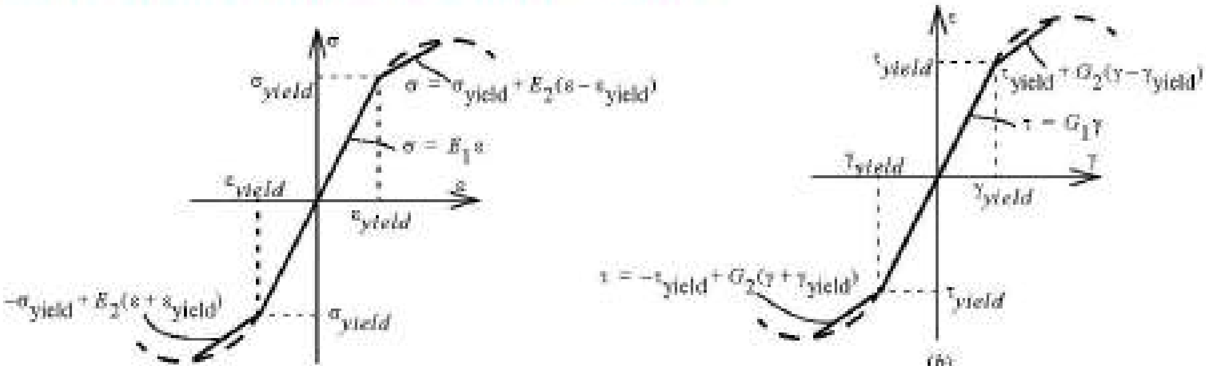


$$\sigma = \begin{cases} \sigma_{yield} & \varepsilon \geq \varepsilon_{yield} \\ E\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} & \varepsilon \leq -\varepsilon_{yield} \end{cases} \qquad \tau = \begin{cases} \tau_{yield} & \gamma \geq \gamma_{yield} \\ G\gamma & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield} & \gamma \leq -\gamma_{yield} \end{cases}$$

- The set of points forming the boundary between the elastic and plastic region on a body, is called the elastic-plastic boundary.

1. On the elastic-plastic boundary the strain must be equal to the yield strain, and stress equal to yield stress.
2. Deformations and strains are continuous at all points including points at the elastic plastic boundary.
3. In beam bending, the location of neutral axis depends material property, geometry, and loading.

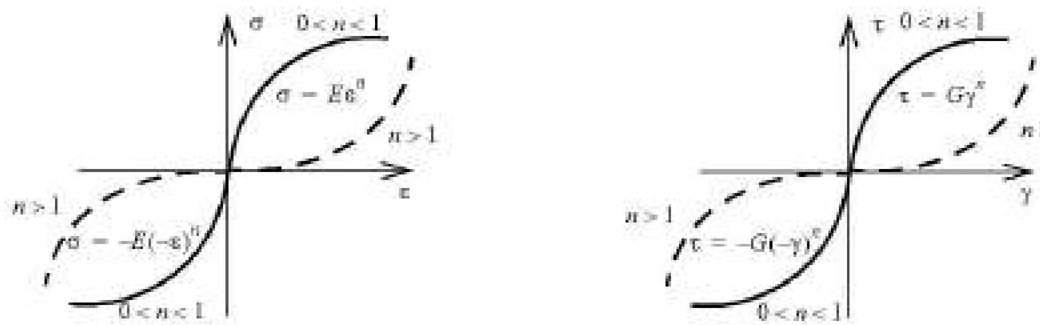
Linear strain hardening material model



(a) (b)

$$\sigma = \begin{cases} \sigma_{yield} + E_2(\varepsilon - \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \\ E_1\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} + E_2(\varepsilon + \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \end{cases}$$

Power Law



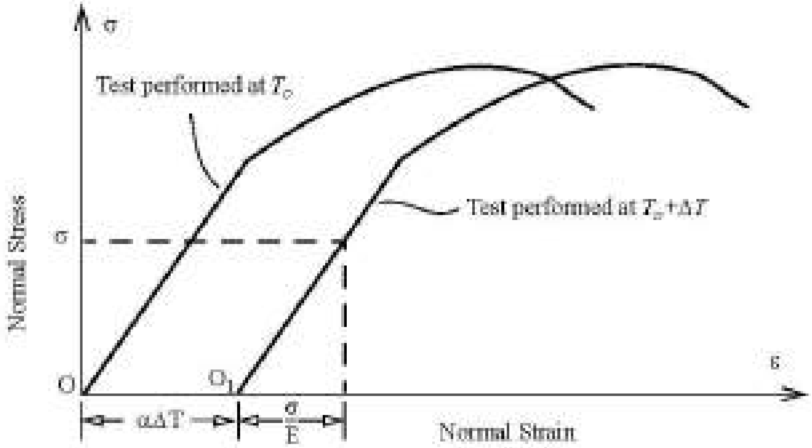
(a) (b)

$$\sigma = \begin{cases} E\varepsilon^n & \varepsilon \geq 0 \\ -E(-\varepsilon)^n & \varepsilon < 0 \end{cases}$$

$n < 1$ —metals and plastics

$n > 1$ —soft rubber, muscles, and organic materials.

Effects of Temperature



$$\varepsilon = \frac{\sigma}{E} + \alpha\Delta T$$

α is linear coefficient of thermal expansion that has units of μ/°F or μ/°C

- No thermal stresses are produced in a homogeneous, isotropic, unconstrained body due to uniform temperature changes.

$$\begin{aligned}
\varepsilon_{xx} &= [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E + \alpha\ \Delta T \\
\varepsilon_{yy} &= [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E + \alpha\ \Delta T \\
\varepsilon_{zz} &= [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E + \underbrace{\alpha\ \Delta T}_{\text{Thermal Strain}} \\
\gamma_{xy} &= \tau_{xy}/G \\
\gamma_{yz} &= \tau_{yz}/G \\
\gamma_{zx} &= \tau_{zx}/G
\end{aligned}$$

Mechanical Strain

C1.4 The stress at a point, material properties, and change in temperature are as given below. Calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, and σ_{zz} (a) assuming plane stress, and (b) assuming plane strain.

$$\sigma_{xx} = 300 \ MPa(C) \qquad \sigma_{yy} = 300 \ MPa(T) \qquad \tau_{xy} = 150 \ MPa$$

$$G = 15 \ GPa \qquad \nu = 0.2 \qquad \alpha = 26.0\mu/^{o}C \qquad \Delta T = 75^{o}C$$

Failure Theories

- A failure theory is a statement on relationship of the stress components to material failure characteristics values.

	Ductile Material	Brittle Material
Characteristic failure stress	Yield stress	Ultimate stress
Theories	1. Maximum shear stress 2. Maximum octahedral shear stress	1. Maximum normal stress 2. Modified Mohr

Maximum shear stress theory

For ductile materials the theory predicts

A material will fail when the maximum shear stress exceeds the shear stress at yield that is obtained from uniaxial tensile test.

The failure criterion is

$$\tau_{max} \le \tau_{yield}$$

$$\boxed{max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) \le \sigma_{yield}}$$

Maximum octahedral shear stress theory (Maximum distortion strain energy or von-Mises criterion)

For ductile materials the theory predicts

A material will fail when the maximum octahedral shear stress exceeds the octahedral shear stress at yield that is obtained from uniaxial tensile test.

The failure criterion is

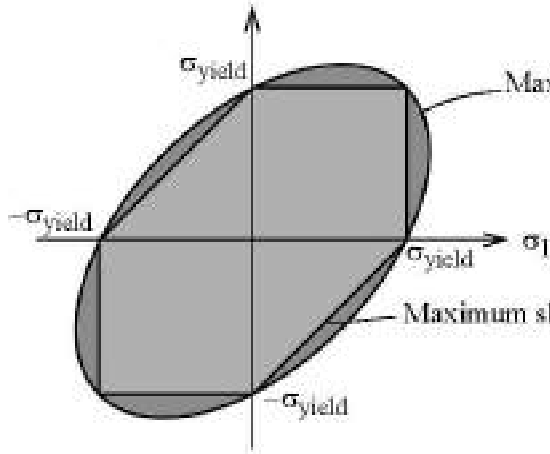
$$\tau_{oct} \le \overline{\tau_{yield}}$$

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le \sigma_{yield}$$

Equivalent von-Mises Stress

$$\boxed{\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}} \qquad \boxed{\sigma_{von} \le \sigma_{yield}}$$

Failure Envelopes for ductile materials in plane stress



Maximum normal stress theory

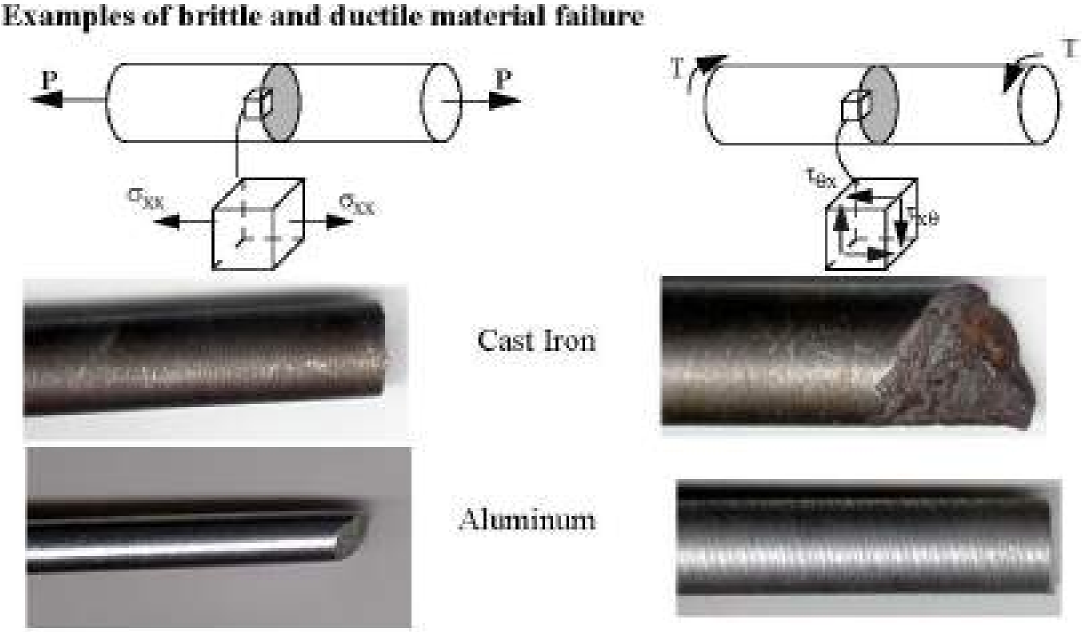
For brittle materials the theory predicts

A material will fail when the maximum normal stress at a point exceed the ultimate normal stress (σ_{ult}) obtained from uniaxial tension test.

$$\boxed{max(\sigma_1, \sigma_2, \sigma_3) \le \sigma_{ult}}$$

- can be used *if principal stress one is tensile* and the *dominant principal* stress.

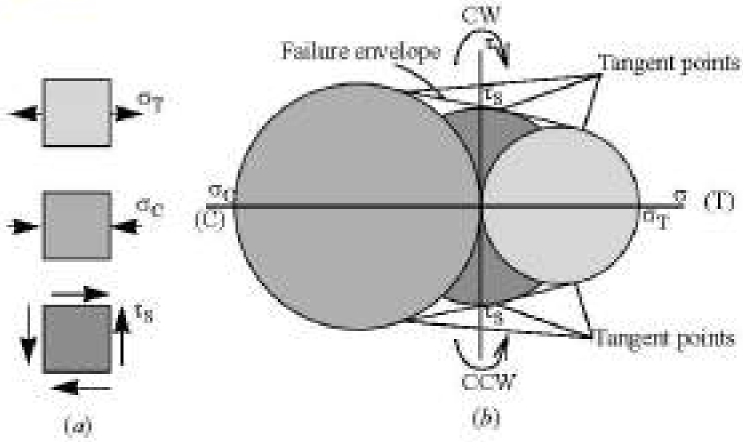
Examples of brittle and ductile material failure



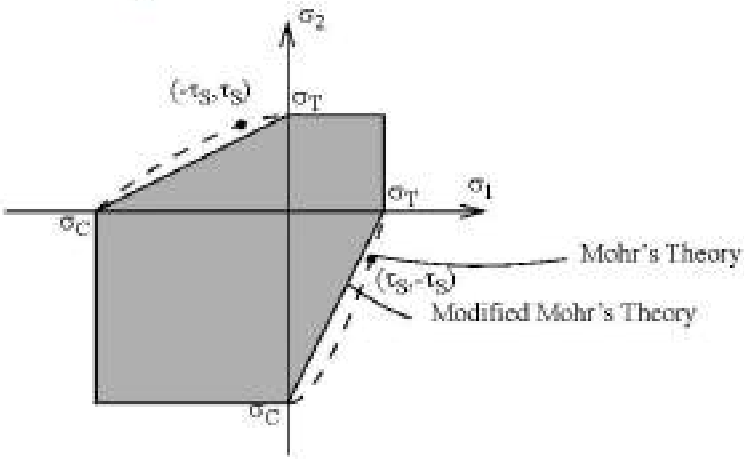
Mohr's theory

For brittle materials the theory predicts

A material will fail if a stress state is on the envelope that is tangent to the three Mohr's circles corresponding to: uniaxial ultimate stress in tension, to uniaxial ultimate stress in compression, and to pure shear.



Modified Mohr's Theory

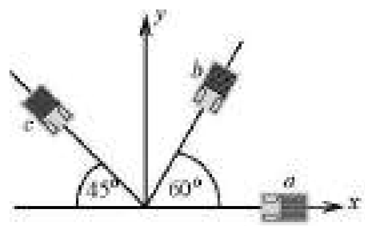


- If both principal stresses are tensile than the maximum normal stress has to be less than the ultimate tensile strength.
- If both principal stresses are negative than the maximum normal stress must be less than the ultimate compressive strength.
- If the principal stresses are of different signs then for the Modified Mohr's Theory the failure is governed by

$$\left|\frac{\sigma_2}{\sigma_C} - \frac{\sigma_1}{\sigma_T}\right| \le 1$$

- σ_C is the magnitude of the compressive strength.

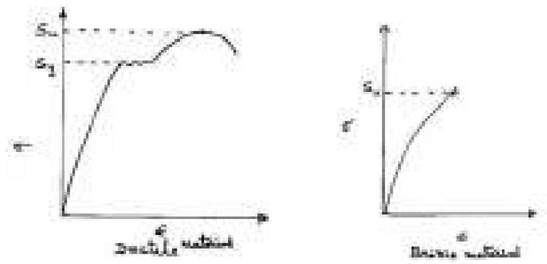
C1.5 On a free surface of aluminum (E = 10,000 ksi, ν = 0.25, σ_{yield} = 24 ksi), the strains recorded by the three strain gages shown below are $\varepsilon_a = -600\ \mu$ in/in, $\varepsilon_b = 500\ \mu$ in/in, and $\varepsilon_c = 400\ \mu$ in/in. By how much can the loads be scaled without exceeding the yield stress of aluminum at the point? Use the maximum shear stress theory.



Theories of Failure

Strength of a material or failure of the material is deduced generally from uni-axial tests from which stress strain characteristics of the material are obtained.

The typical stress-strain curves for ductile and brittle materials are shown below.



Material Strength parameters are S_y OR S_u

Theories of Failure

In the case of multidimensional stress at a point we have a more complicated situation present. Since it is impractical to test every material and every combination of stresses σ_1, σ_2, and σ_3, a failure theory is needed for making predictions on the basis of a material's performance on the tensile test., of how strong it will be under any other conditions of static loading.

The "theory" behind the various failure theories is that ***whatever is responsible for failure in the standard tensile test will also be responsible for failure under all other conditions of static loading.***

Theories of Failure

The microscopic yielding mechanism in ductile material is understood to be due to relative sliding of materials atoms within their lattice structure. This sliding is caused by shear stresses and is accompanied by distortion of the shape of the part. Thus the yield strength in shear S_{sy} is strength parameter of the ductile material used for design purposes.

Generally used theories for Ductile Materials are:

•Maximum shear stress theory

•Maximum distortion energy theory.

(von Mises-Hencky's theory).

Theories of Failure

The Maximum - Shear - Stress Theory

The Maximum Shear Stress theory states that *failure occurs when the maximum shear stress from a combination of principal stresses equals or exceeds the value obtained for the shear stress at yielding in the uniaxial tensile test.*

At yielding, in an uni-axial test, the principal stresses are

$\sigma_1 = S_y$; $\sigma_2 = 0$ and $\sigma_3 = 0$.

Therefore the shear strength at yielding

$S_{sy} = [\sigma_1 - (\sigma_2 \text{ or } \sigma_3 = 0)]/2$. Therefore $\mathbf{S_{sy} = S_y/2}$

Theories of Failure

(Maximum Shear Stress theory)

To use this theory for either two or three-dimensional static stress in homogeneous, isotopic, ductile materials, first compute the three principal stresses (σ_1, σ_2, σ_3) and the maximum shear stress τ_{13} as

$$\tau_{\max} = \frac{(\sigma_1 - \sigma_2)}{2} \quad = \quad \frac{(\sigma_{p\max} - \sigma_{p\min})}{2}$$

Then compare the maximum shear stress to the failure criterion.

$$\tau_{\max} \le S_{sy} \quad \text{OR} \quad \frac{(\sigma_{p\max} - \sigma_{p\min})}{2} \le S_{sy}$$

The safety factor for the maximum shear-stress theory is given by

$$N = \frac{S_{sy}}{\tau_{\max}}$$

Theories of Failure

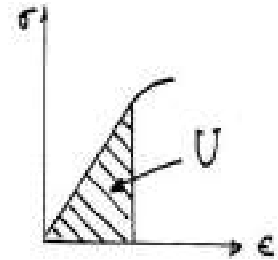
Distortion-Energy Theory OR The von Mises - Hencky Theory

It has been observed that a solid under hydro-static, external pressure (e.g. volume element subjected to three equal normal stresses) can withstand very large stresses.

When there is also energy of distortion or shear to be stored, as in the tensile test, the stresses that may be imposed are limited.

Since, it was recognized that engineering materials could withstand enormous amounts of hydro-static pressures without damage, it was postulated that a given material has a definite limited capacity to absorb energy of distortion and that any attempt to subject the material to greater amounts of distortion energy result in yielding failure.

Total Strain Energy: Assuming that the stress-strain curve is essentially linear up to the yield point, we can express the total strain energy at any point in that range as.



$U = \frac{1}{2}\sigma\varepsilon$ (a)

Extending this to three dimensional case

$U = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$ (b)

where σ_1, σ_2, and σ_3 are principal stresses and ε_1, ε_2 and ε_3 are principal strains.

Expressing strains in terms of stresses as.

$$\varepsilon_1 = 1/E\,[\,\sigma_1 - \nu\,(\sigma_2 + \sigma_3)\,]$$
$$\varepsilon_2 = 1/E\,[\,\sigma_2 - \nu\,(\sigma_1 + \sigma_3)\,] \quad \text{(c)}$$
$$\varepsilon_3 = 1/E\,[\,\sigma_3 - \nu\,(\sigma_2 + \sigma_1)\,]$$

U, the total energy can be written by substituting (c) into (b)

$$U = (1/2E)\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \text{(d)}$$

Let U_h be energy due to volume change and U_d be energy due to distortion. Then we can express each of the principal stresses in terms of hydrostatic component (σ_h), common to all the faces of volume element and distortion component (σ_{id}) that is unique to each face.

$$U = U_h + U_d \quad \text{(e)}$$
$$\sigma_1 = \sigma_h + \sigma_{1d}$$
$$\sigma_2 = \sigma_h + \sigma_{2d} \quad \text{(f)}$$
$$\sigma_3 = \sigma_h + \sigma_{3d}$$

Adding the three principal stresses, gives

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h + (\sigma_{1d} + \sigma_{2d} + \sigma_{3d}) \quad \text{(g)}$$

For volumetric change with no distortion, the terms in the bracket of eqn (g) must be zero. Thus, we have

$$\sigma_h = 1/3\,(\sigma_1 + \sigma_2 + \sigma_3) \quad \text{(h)}$$

Now, U_h can be obtained by replacing principal stresses in eqn (d) as

$$U_h = (1/2E)\,[\sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu\,(\sigma_h^2 + \sigma_h^2 + \sigma_h^2)]$$
$$= 3/2\,(1-2\nu)/E\,\sigma_h^2 \text{ -------(i)}$$

Substituting eqn (h) into eqn (i), we obtain

$$U_h = (1-2\nu)/E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \text{(j)}$$

$U = U_h + U_d$

<u>Distortion Energy</u>

$$U_d = U - U_h$$

substituting (j) and (d) into (k), we obtain

$U_d = (1 + \nu)/3E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1]$

In uniaxial stress-state at yield

$$\sigma_1 = S_y;\ \sigma_2 = \sigma_3 = 0.$$

Therefore energy of distortion in uni axial stress state

$U_{du} = (1 + \nu)/3E\ S_y^2 \qquad (m)$

Failure criterion according to distortion energy is

$(1 + \nu)/3E\,S_y^2 = (1 + \nu)/3E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]$

$S_y = [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]^{1/2}$

Theories of Failure

"<u>Distortion energy theory</u> states <u>*that failure by yielding under a combination of stresses occurs when the energy of distortion equals or exceeds the energy of distortion in the tensile test when the yield strength is reached.*"</u>

According to theory failure criteria is

$S_y = [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1]^{1/2}$

For two dimensional stress state ($\sigma_2 = 0$), the equations reduces to

$S_y = [\sigma_1^2 + \sigma_3^2 - \sigma_3\sigma_1]^{1/2}$

Theories of Failure

It is often convenient in situations involving combined tensile and shear stresses acting at a point to define an effective stress that can be used to represent the stress combination.

<u>The von-Mises effective stress</u> (σ_e) also sometimes referred to as <u>equivalent stress</u> is defined as the uniaxial tensile stress that would create the same distortion energy as is created by the actual combination of applied stresses.

$$\sigma_e = \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]^{1/2}$$

$$\sigma_e = \left[\sigma_1^2 + \sigma_3^2 - \sigma_3\sigma_1\right]^{1/2}$$

In terms of applied stresses in coordinate directions

$$\sigma_e = \left[\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2\right]^{1/2}$$

$$\text{Safety factor } N = \frac{S_y}{\sigma_e}$$

<u>Static Failure Theories for Brittle Materials</u>

- Brittle materials fracture than yield.
- <u>Brittle Fracture</u> in tension is considered to be due to normal tensile stress alone and thus the maximum normal-stress theory is applicable.
- Brittle fracture in compression is due to some combination of normal compressive stress and shear stress.

<u>Even and Uneven Materials</u>

•Some wrought materials, such as fully hardened tool steel, can be brittle. These materials tend to have compressive strength equal to their tensile strengths . They are called <u>EVEN</u> materials.

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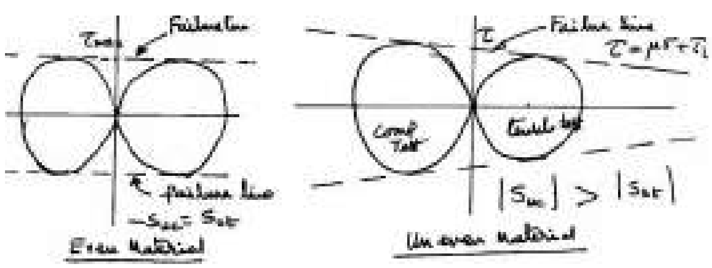
•Many cast materials, such as gray cast iron, are brittle but have compressive strengths much greater than their tensile strengths. These are called <u>UNEVEN</u> materials.

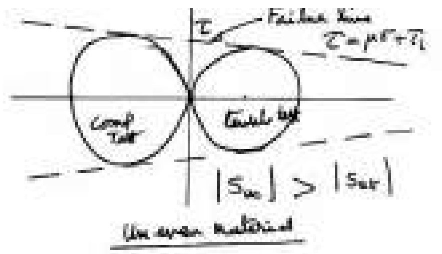
- For uneven materials; tensile strength is due to the presence of microscopic flaws in the castings, which when subjected to tensile loading, serve as nuclei for crack formation.
- when subjected to compressive stress, these flaws are pressed together, increasing the resistance to slippage from shear stresses.
- Gray cast irons typically have compressive strengths 3 to 4 times their tensile strengths and ceramics have even larger ratios.
- Another characteristics of some cast, brittle materials is that their shear strength can be greater than their tensile strength, falling between their compressive and tensile strengths.

Mohr's circles for both compression and tensile tests of an <u>even</u> and <u>uneven</u> materials are shown below.



The lines tangent to these circles constitute failure lines for all combinations of applied stress between the two circles. The area enclosed by the circles and the failure lines represent a safe zone.

In the case of even material, the failure lines are independent of the normal stresses and are defined by the maximum shear strength of the material. This is consistent with the maximum shear stress theory.



- For the uneven material, the failure lines are a function of both normal stresses and shear stresses. For compressive regime, as the normal stress component becomes increasingly negative (i.e. more compression) the material's resistance to shear stress increases.
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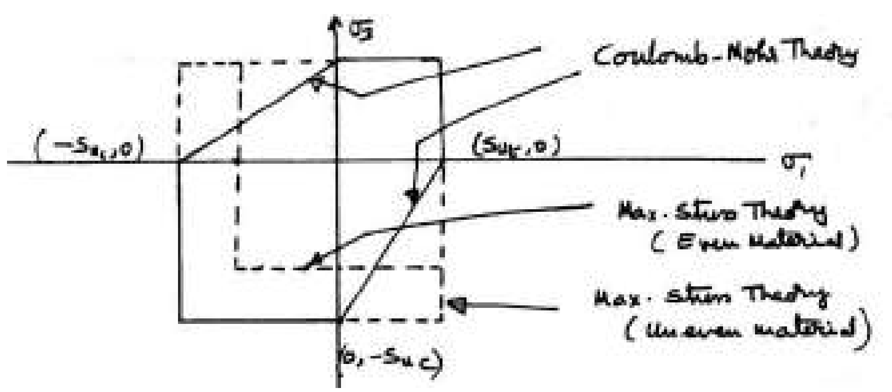
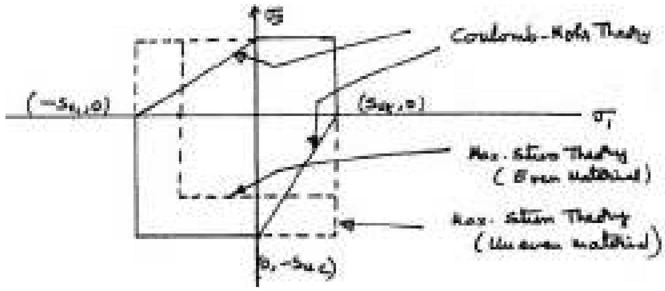
- The interdependence between shear and normal stress is confirmed by experiment for cases where the compressive stress is dominant, specifically where the principal stress having the largest absolute value is compressive.
- However, experiments also show that in tensile-stress-dominated situations with uneven, brittle materials, failure is due to <u>tensile stress alone</u>. The shear stress appears not to be a factor in uneven materials if the largest absolute value is tensile.

<u>Maximum Normal Stress Theory</u>

The maximum normal stress theory, shown for even materials could be used as the failure criterion for brittle materials in static loading if compressive and tensile strengths were equal (even material).

The maximum-normal stress theory envelope for an uneven material as the asymmetric square of half-dimensions S_{ut}, - S_{uc} is also shown. This failure envelope is only valid in the first and third quadrants as it does not account for the interdependence of normal and shear stresses which affects second and fourth quadrants.

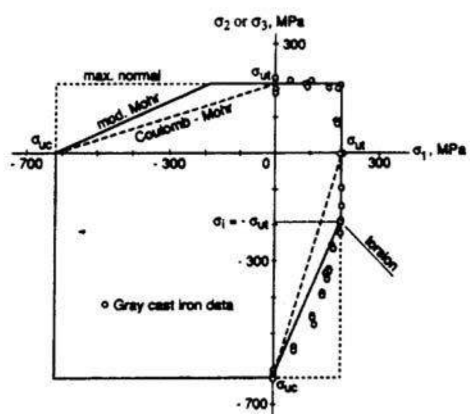




Coulomb-Mohr theory

The coulomb-Mohr envelope attempts to account for the interdependence by connecting opposite corners of these quadrants with diagonals.

The Figure shows some gray cast-iron experimental test data superposed on the theoretical failure envelopes.

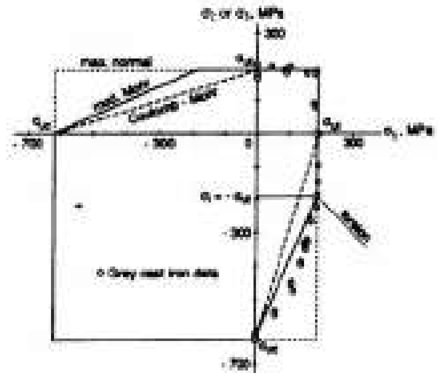


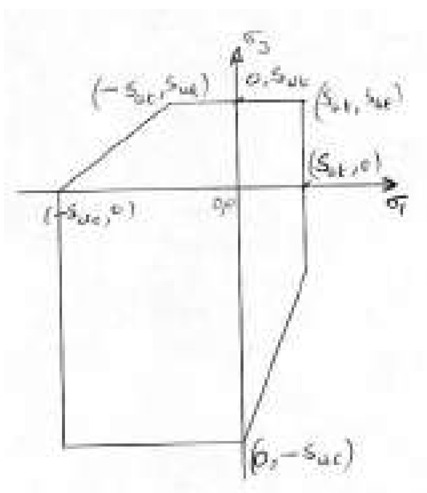
- The failures in the first quadrant fit the maximum normal-stress theory line.
- The failures in the fourth quadrant fall **inside** the maximum normal-stress line (indicating its unsuitability)
- Also experimental data fall **outside** the Coulomb-Mohr line (indicating its conservative nature).

This observation leads to a modification of the Coulomb-Mohr theory to make it better fit the observed data.

The actual failure data in the above figure follow the even material' maximum normal stress theory envelop down to a point S_{ut}, $-S_{ut}$ below the σ_1 axis and then follow a straight line to 0, $-S_{uc}$. The set of lines shown by a solid line is the **modified-Mohr failure theory envelop.**

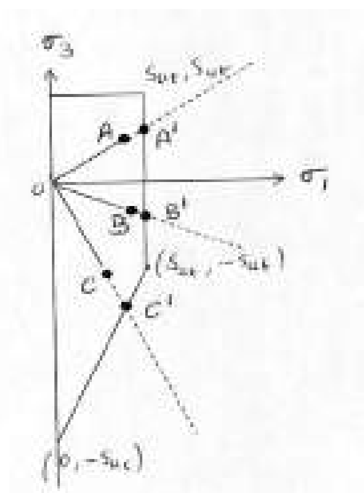
It is the preferred failure theory for uneven, brittle materials in static loading.





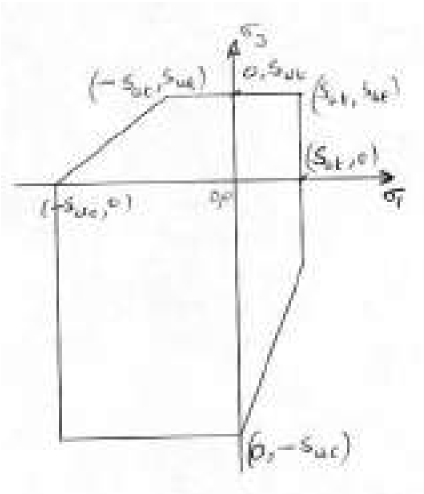
If the 2-D principal stresses are ordered $\sigma_1 > \sigma_3$, $\sigma_2 = 0$, then only the first and fourth quadrants need to be drawn as shown in Figure

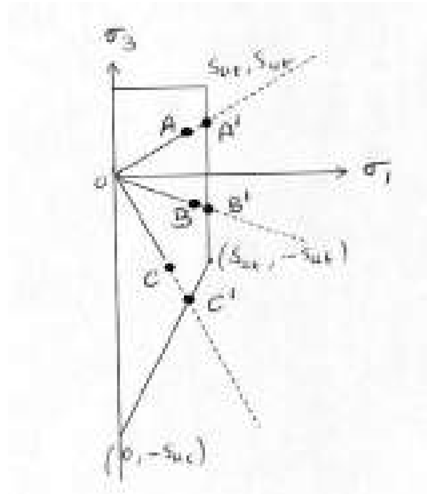
the figure depicts three plane stress conditions labeled A, B, and C.



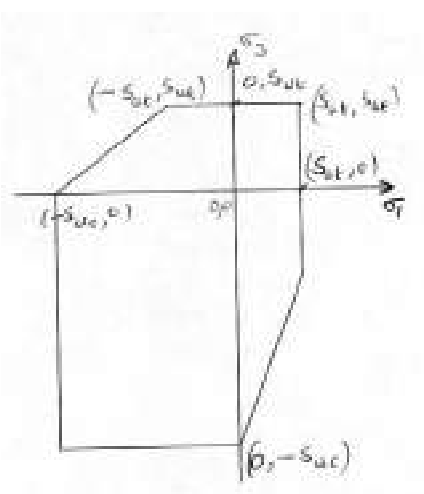
Point A represents any stress state in which the two non zero principal stresses σ_1, σ_3 are positive. Failure will occur when the load line OA crosses the failure envelop at A' The safety factor for this situation can be expressed as

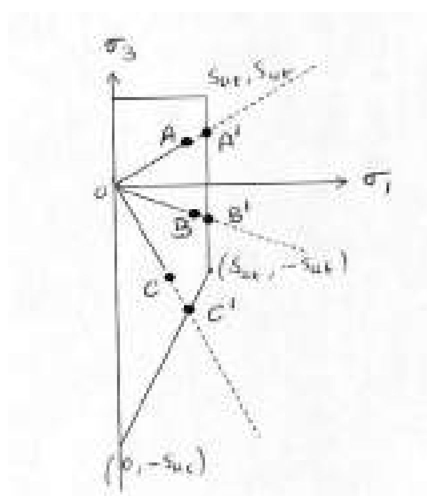
$$N = S_{ut}/\sigma_1$$





If the two nonzero principal stresses have opposite sign, then two possibilities exist for failure, as depicted by points B and C. The only difference between these two points is the relative values of their two stress components σ_1, and σ_3. The load line OB exits the failure envelop at B' above the point(S_{ut},-S_{ut}) and the safety factor for this case is the same as the previous equation.





If the stress state is as depicted by point C, then the intersection of the load line OC and the failure envelop occurs at C' below the point (S_{ut}, -S_{ut}). The safety factor can be found by solving for the intersection between the load line OC and the failure line and is given by

$$N = \frac{S_{ut} S_{uc}}{S_{uc}\sigma_1 - S_{ut}(\sigma_1 + \sigma_3)}$$

If the stress state is in the fourth quadrant both of these equations should be checked and the resulting smaller safety factor used.